On The Arithmeticity of FAL's

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Abstract

Fully augmented links "often admit hyperbolic structures that can be explicitly described in terms of combinatorial information coming from their respective link diagrams" [1]. Within these hyperbolic structures exist measurable geodesic lengths that allow us to realize information about whether a FAL is arithmetic or not. Being able to deduce this enables us to gauge how to approach other questions we have regarding the given link. Much more is known about arithmetic FAL's. The goal of this research is to determine a way to compute the geodesic length of FAL's with circle packings nested between two concentric circles, the smaller of which is the unit circle. Given these lengths, we can classify the links as arithmetic or non-arithmetic via a comparison to work in [1] and thus gain more clarity on how to preform other computations around a given link.

Knots are closed curves in three dimensions and links are a collection of knots that may be interconnected but not intersected. We will focus on a class of links called fully augmented links (FAL's), which are links that have been strategically simplified to reduce the number of twist regions. This simplification, called augmentation, takes the diagram of a knot or link and places a trivial component called a crossing circle around every twist region. Twists with an odd number of crossings are reduced to a single crossing within the crossing circle, and twits with an even number of crossings are completely unwound. The link that remains is fully augmented.

We can realize a geometric representation of any given fully augmented link via a diagram called a circle packing. To determine a circle packing, we must be aware of the fact that fully augmented links are hyperbolic links, which are a class of links whose complement is a complete hyperbolic manifold [2]. This complement has two important properties [1]:

1. M_f decomposes into two identical ideal totally geodesic polyhedra, $P\pm$. [4]
2. Intersecting a crossing disk with the projection plane creates two half disks. If we peel these two half disks apart, then four shaded faces are produced. Each of these four shaded faces is an ideal hyperbolic triangle, two in P^+ and two in P^-.

The manifold can then be thought of as a collection of these polyhedra glued together via specific instructions [2]. To obtain the circle packing, we perform a cell decomposition on our fully augmented link.

1. Cut the hyperbolic link complement along the plane of projection. This gives us P^+ and P^-, the two ideal geodesic polyhedra as mentioned above. These polyhedra are isometric and symmetric about the projection plane. Note that half of each crossing circle is in each region.
2. For both P^+ and P^-, slice vertically ("pita cut") and flatten the crossing disks.
3. Shrink all link compliments into ideal vertices.
4. Transform all faces that form from the projection plane into circles with the same tangencies as the cell decomposition.

We then have a circle packing for our given FAL. [5]

Proposition 5.2 in [1] will reveal information that we can draw upon to derive our proof. The proposition and a summary of the proof is as follows:

Let M_n be a hyperbolic link compliment. If $n \geq 7$, then M_n and all of its polyhedral partners are non-arithmetic.

Proof. The proof for 5.2 is restricted to hyperbolic link compliments M_n with circle packings that contain two concentric circles W_{n+1} and W_{n+2} with n circles nested between both, tangent to both W_i and both adjacent n-circles.
Allow γ^+ to be the vertical line segment through the origin that is the common perpendicular to W_{n+1} and W_{n+2}. The length of γ^+ can be found utilizing the radii of the W_i.
Thus

$$\ell(\gamma^+) = \ln\left(\frac{\csc\left(\frac{\pi}{n}\right)+1}{\csc\left(\frac{\pi}{n}\right)-1}\right)$$

is the hyperbolic length of γ^+.
Since the circle packing was formed from one fundamental region of the polyhedra, we must double our equation in order to arrive at the complete geodesic length γ:

$$\ell(\gamma) = 2\ln\left(\frac{\csc\left(\frac{\pi}{n}\right)+1}{\csc\left(\frac{\pi}{n}\right)-1}\right).$$

When $n \geq 15$, the geodesic length is less then 0.862554627 and by [3] restated in Theorem 5.1 in [1], M_n is non-arithmetic. When $7 \leq n \leq 15$, we compare the geodesic length to a given table of values found by Walter D. Neumann and Alan W. Reid [3] to determine the arithmeticity of M_n, and when $n \leq 7$, M_n is arithmetic. $\square$

Here we will expand on Proposition 5.2 to find a formula for the hyperbolic length of a geodesic when the inner concentric circle is the unit circle, and thus determine the arithmeticity of the hyperbolic link compliment.

Proposition 1. *Let U be the unit circle and let C be any circle centered at the origin with a radius $R > 1$. If n non-overlapping circles are tangent to both U and C, then $R \leq \frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}$.*

Proof. Suppose we have two concentric circles, the larger of which C has a radius $R > 1$ and the smaller of which is the unit circle U.
For all circle packing's of n circles between C and U, we will require all n circles to be tangent to both C and U, thus sharing a common radius r.
Notice that we can define r in terms of R:

$$R = 1 + 2r$$
$$R - 1 = 2r$$
$$\frac{R-1}{2} = r$$

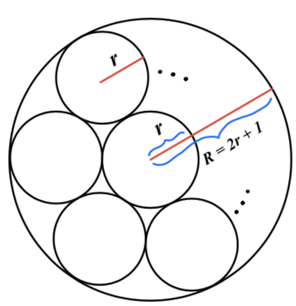

Figure 1: Radii in a concentric circle packing.

Next, notice that since the centers of the n circles can be evenly spaced around U, we can form a regular n-gon centered at $(0,0)$ with vertices at the center of each of our n circles.

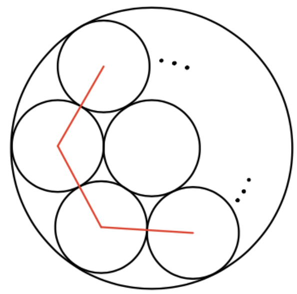

Figure 2: A regular n-gon with vertices at the centers of the n circles.

From this n-gon we can solve for the distance between two vertices, which we will designate l.
To find l for a given n, notice that the n-gon can be broken down into n isosceles triangles with symmetrical sides of length $r+1$ and a common angle $\lambda = \frac{2\pi}{n}$. The third, shorter side has length l.
We can find two right triangles within each isosceles, with an angle $\theta = \frac{\pi}{n}$, half of that of λ.

We can then solve $\sin(\theta) = \frac{l}{2(r+1)}$ in terms of l:

$$\sin(\theta) = \frac{l}{2(r+1)}$$
$$\sin\left(\frac{\pi}{n}\right) = \frac{l}{2(r+1)}$$
$$l = \sin\left(\frac{\pi}{n}\right) \cdot 2(r+1)$$
$$l = \sin\left(\frac{\pi}{n}\right) \cdot 2\left(\frac{R-1}{2} + 1\right)$$
$$l = \sin\left(\frac{\pi}{n}\right) \cdot (R+1)$$

We can now realize the implications of three different kinds of circle packings.

1. Consider when $2r = l$.
   The length of a side of a regular n-gon is thus equal to twice the radius of the n-circles. This realizes the fact that the n-circles are tangent to each other as well as to C and U in the packing.
   We can then then rearrange our equation for l to be in terms of R and see that R will be the upper bound on the radius of the outer circle when we restrict the n-circles in the packing from overlapping.

$$l = \sin\left(\frac{\pi}{n}\right) \cdot (R+1)$$
$$2r = \sin\left(\frac{\pi}{n}\right) \cdot (R+1)$$
$$R - 1 = \sin\left(\frac{\pi}{n}\right) \cdot (R+1)$$
$$R = \frac{\sin\left(\frac{\pi}{n}\right) + 1}{1 - \sin\left(\frac{\pi}{n}\right)}$$

   Thus, we have found an equation for the upper bound on R for n circles when they are tangent to each other and C and U.

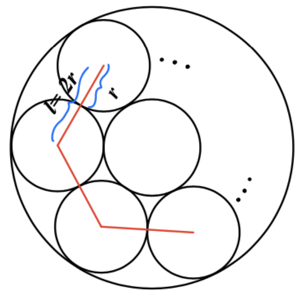

Figure 4: When $l = 2r$ the n circles are tangent to each other and to C and U.

2. Consider when $2r > l$. The length of a side of a regular n-gon is thus *less* then twice the radius of the n-circles. This realizes the fact that that the n-circles are *overlapping* as well as tangent to C and U in the packing.
   We can then rearrange our equation for l to be in terms of R and see that R must be strictly greater then a certain value.

$$R > \frac{\sin\left(\frac{\pi}{n}\right) + 1}{1 - \sin\left(\frac{\pi}{n}\right)}$$

   Thus, we have found that R will be strictly greater then $\frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}$, verifying that $R = \frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}$ is the upper bound for n circles to be tangent.

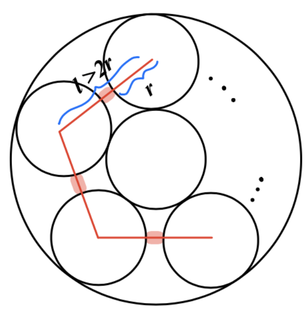

Figure 5: When $l < 2r$ the n circles are not touching each other, but are still tangent to C and U.

3. Consider when $2r < l$. The length of a side of a regular n-gon is thus *greater* then twice the radius of the n-circles. This realizes the fact that that the n-circles are *not* tangent *nor* overlapping, but still are tangent to C and U in the packing.
   We can then rearrange our equation for l to be in terms of R and see that R must be strictly less then a certain value.

$$R < \frac{\sin\left(\frac{\pi}{n}\right) + 1}{1 - \sin\left(\frac{\pi}{n}\right)}$$

   Thus, we have found that R will be strictly less then $\frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}$, and for any R lower the n-circles will not touch.

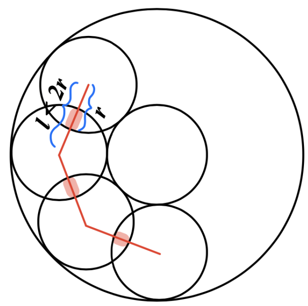

Figure 6: When $l > 2r$ the n circles are overlapping and are still tangent to C and U.

We can thus conclude that $R = \frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}$ is an upper bound on the radius of C for n circles, and that these circles must be tangent in order to determine R.

$\square$

Notice that from our equation for R we can solve for the hyperbolic length of a vertical line segment through the origin of U and C, thus making it a common perpendicular of the two circles (just as in Proposition 5.2). To solve for the hyperbolic length γ, we can use the equation

$$\ell(\gamma) = 2\ln\left(\frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}\right)$$

[1].
This leads naturally to a another conclusion.

Theorem 1. *If M_n is a FAL such that it's circle packing contains at least two disjoint circles with n circles tangent to both, then M_n contains a geodesic with a length γ at most*

$$\ell(\gamma) \leq 2\ln\left(\frac{\sin(\frac{\pi}{n})+1}{1-\sin(\frac{\pi}{n})}\right).$$

Proof. Suppose M_n is a fully augmented link such that it's circle packing contains at least two disjoint circles U and C with n circles tangent to both.
Notice that when two planes are disjoint in $\mathbb{H}^3$ they possess a common perpendicular ℓ with endpoints p and q.

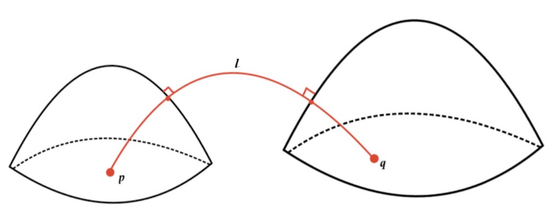

Figure 7: Two disjoint planes with a common perpendicular l and endpoints p and q.

We can then use a hyperbolic isometry to send q to infinity such that ℓ becomes a vertical line ℓ' perpendicular to the plane.
Since U and C are perpendicular to ℓ', they become concentric circles centered at p'.
The ideal boundary circles of U and C are thus concentric circles centered at p'.

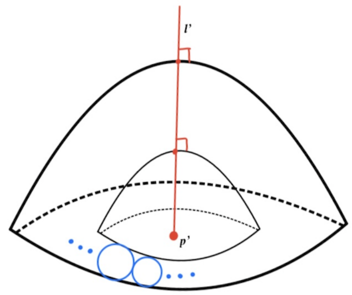

Figure 8: Concentric ideal boundary circles centered at *p'*.

We now use a dilation (another hyperbolic isometry) centered at p' to make U have a radius of 1.

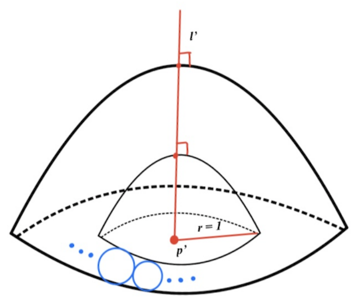

Figure 9: Concentric ideal boundary circles centered at *p'* after an isomorphism to set the radius of U to 1.

Notice we now have a circle packing for M that allows for us to apply Proposition 1 and solve for the hyperbolic length of it's geodesic.
Thus, for any FAL such that it's circle packing contains at least two disjoint circles with n circles tangent to both, we can utilize hyperbolic isometries to transform the circle packing to one such that there are two concentric circles, the smaller of which the unit circle, with n-circles tangent to both. From this form we can compute the length of it's geodesic and determine it's arithmeticity. $\square$

From the above conclusions we are able to easily arrive at a further result. Before doing so, we must take note of a result in [3] that states, "If M [a hyperbolic link compliment] contains a geodesic of length less than 0.862554627, then M must be non-arithmetic." We proceed with the following corollary:

Corollary 1. *Let C and U be disjoint circles in the circle packing of M_n, the hyperbolic link compliment of some FAL. Suppose there are n circles in the packing that are tangent to both C and U. If $n \geq 15$, then M_n is not arithmetic.*

Notice that when $n \geq 15$,

$$\begin{aligned}
\ell(\gamma) &\leq 2\ln(R) \\
&\leq 2\ln\left(\frac{\sin\left(\frac{\pi}{15}\right)+1}{1-\sin\left(\frac{\pi}{15}\right)}\right) \\
&\leq 2\ln\left(\frac{\sin\left(\frac{\pi}{15}\right)+1}{1-\sin\left(\frac{\pi}{15}\right)}\right) \\
&= 0.843960770 < 0.862554627
\end{aligned}$$

and M_n is non-arithmetic.
Furhter, $\ell(\gamma) = \ln\left(\frac{\sin\left(\frac{\pi}{n}\right)+1}{1-\sin\left(\frac{\pi}{n}\right)}\right)$ is a decreasing function since it's derivative is

$$\ell'(\gamma) = -\frac{2\pi\sec\left(\frac{\pi}{n}\right)}{n^2}$$

which is negative.
This verifies that for any $n \geq 15$, the value of $\ell(\gamma)$ will continue to decrease and be less then 0.862554627. We thus see that given the number of circles in the packing tangent to both C and U, we can compute the geodesic length of the FAL and further categorize it as arithmetic or not.

To better understand the types of FAL's discussed in this paper, it is best to consider a diagrammatic example. Below is a fully augmented link diagram with $n \geq 15$ crossing circles. These circles will represent the points of tangency of the circles formed in the circle packing, and the link chain that forms from the augmentation will become the n circles tangent to two concentric circles U and C and eachother.

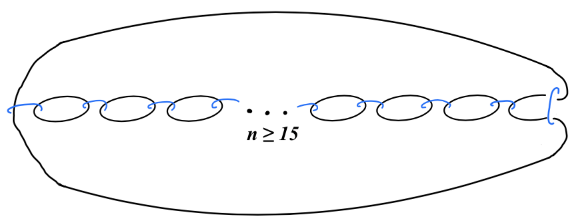

Figure 10: Non-arithmetic FAL by Corollary 1.

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