Method to Generate b-Prime Fully Augmented Links

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Abstract

In a effort to enumerate fully augmented links that are not belted-sum decomposable (b-prime), this work will present an operation, called complete augmentation, on prime links that will produce all b-prime fully augmented links (FAL). A lemma proven by Jorge Calvo in 1985 will be vital in order to prove all completely augmented prime links will result in a b-prime FAL.

1 Introduction

This paper will build off the work of Jorge Alberto Calvo in his work to enumerate knots[1], the work of Morgan et. al[2] and Ransom[3] . This will all be to prove a new construction of fully augmented links that will produce only belt-sum prime fully augmented from prime links.

To begin, a belted sum is an operation introduced by Colin Adams in 1985 [4] is the joining of two links that both contain trivial components. The links are then cut along the disks bounded by the trivial components. This allows the two links to be glued together along those components to create one link. This operation is seen in Fig 1. It has been shown that belted sum maintains hyperbolic character and that the volume of the belted sum is equal to the sum of the original links' volumes[2, 4].

Next how to augment a link must be discussed. Purcell describes the following method: Any prime alternating link can be augmented by adding a trivial component around a two strand twist. Then by removing every full twist we achieve the fully augmented link [5]. That is given a link L, then $\mathcal{A}_L$ is the corresponding fully augmented link. Adams proved that if a hyperbolic link is fully augmented then the resulting link is also hyperbolic[4].

2 Results

First we must define a universe that will clearly provide the information about an FAL

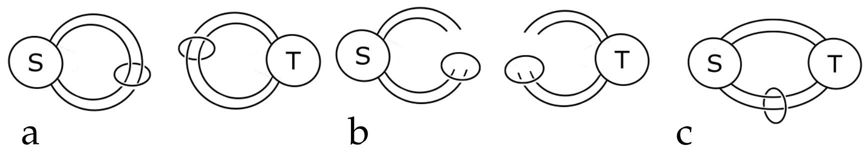

Figure 1: These are the three steps of a belted sum composition. The left figure shows two links with a trivial component. In the middle shows the links cut along the disk bounded by the trivial components. The right figure shows the two links to be glued together along those components to create one link.

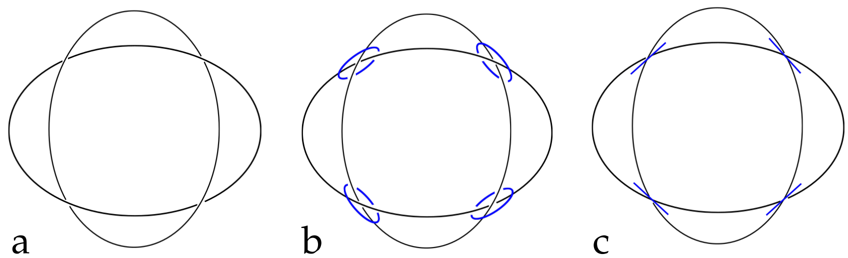

Figure 2: On the left is a link, L. The middle and right figures are the corresponding FAL and FAL in universe, $p(L)$

Definition 2.1. An *fully augmented link universe* of an FAL, $L \in \mathbb{R}^3$, is the image, $p(L) \in \mathbb{R}^2$. A crossing circle in L will be a *crossing arc* in $p(L)$. See Figure 2

Notice that crossing are not recorded in this universe. The ”under” or ”over” strand of a crossing is not necessary to know. Next a tangle in the universe must be defined

Definition 2.2. A *m-tangle* in a FAL universe is the intersection of the universe. $p(L)$, with a disk in the plane, $D \in \mathbb{R}^2$. The boundary of a tangle is denoted ∂D and $\partial D \cap p(L)$ consists of exactly m points *on the knot segments* of $p(L)$. Note that these can be crossings between knot segments and crossing arcs. A *composing tangle* in the FAL universe is a 4-tangle T such that two of the points in $L \cap \partial T$ lie on the same crossing arc.

It is worth noting that it is not allowed for a single point of a crossing arc to lie on the boundary of a tangle. Either the full arc lies on the boundary or it does not intersect. It is useful to this work to define b-prime FAL in terms of tangles, rather than in terms of belted sums. This leads to the following definition.

Definition 2.3. A *b-prime fully augmented link* is a FAL such that there does not exist a non-trivial composing tangle.

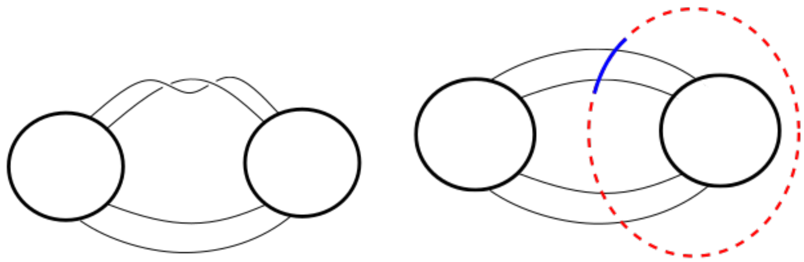

Figure 3: On the left is an example of a non-trivial flype orbit, and the right shows an examples of a composing tangle. This is to show that augmenting a non-trivial flype orbit will create a non-trivial composing tangle.

This notion of non-trivial composing tangles is analogous to a non-trivial flype orbit. A visual representation of this is in Fig, 2. B-prime FAL are defined in this way in order to correspond to Calvo's definition of prime knots [1]. He proved a useful lemma that will be a basis for the main result of this work:

Lemma 2.1. In a prime knot universe, each crossing generates at most one non-trivial flype orbit. [1]

Rewriting this in the terms of fully augmented links it follows that in a knot universe, each crossing can have a trivial component placed around it in at most one direction to generate a non-trivial composing tangle. This means, with a choice of how we place the trivial component around each crossing, there is always a way to not generate a non-trivial composing tangle. This will be useful as the following operation will provides choice about how to augment each crossing in a link to produce an FAL.

Definition 2.4. The *complete augmentation* of a prime link is the addition of a crossing circle around each crossing in such a way to not create a composing tangle

This is a different but similar operation to Purcell's way to form fully augmented links [5]. This begs the question, does a complete augmentation create an FAL? Because the original link is prime, the resulting FAL cannot be split or connect sum composite. Additionally the knot must be twist reduced and have at least two twist regions. So this way to augment will result in a fully augmented link. Note that the crossing arcs that lie on a twist region with more than one crossing must all lie in the same direction, following the length of the twist. This case and if one lies perpendicular are shown in fig. **??**. Now with this operation defined, the main result of this paper can be achieved.

Theorem 2.1. A complete augmentation on a link, L, will result in a b-prime FAL, $p(L)$, if and only if L is prime.

Proof. First if L is prime.

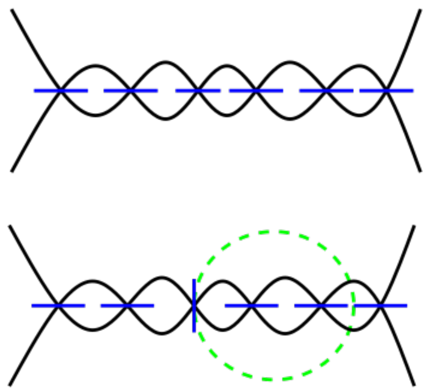

Figure 4: The top diagram show the correct complete augmentation of a twist region with more than 1 crossing. The bottom diagram shows that if any of the crossing are augmented in the other direct then the resulting FAL will not be b-prime.

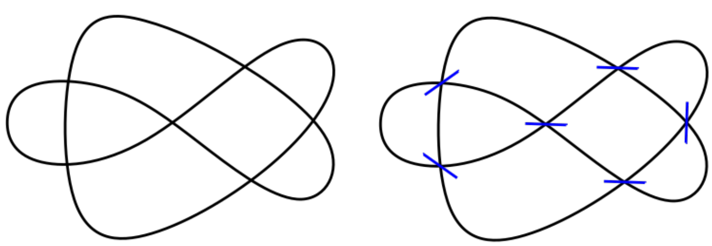

Figure 5: A diagram of the 6_2 knot is show on the left, with the complete augmentation of it on the right. The FAL arising from 6_2 is b-prime.

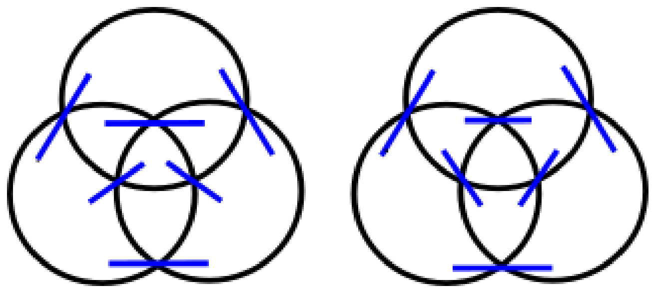

Figure 6: An examples of two of the many ways to completely augment the link L6a4

This will be a proof by contradiction, assuming that the resulting FAL is not b-prime. Therefore there exists a non-trivial composing tangle within $p(L)$. Because L is prime and using Calvo's lemma 2.1, L has been augmented such that all crossing arcs do not bound a non-trivial composing tangle. This is true regardless of the number of crossings in the twist region. This is a contradiction so $p(L)$ must be b-prime.

Now let's consider the converse and suppose that $p(L)$ is b-prime. As a proof by contradiction, assume that L is not prime. This would mean that L is a connect sum, or equivalently that there is a non-trivial 2-tangle in L. Augmenting L in such a way to create $p(L)$ will not affect the non-trivial 2-tangle in L. This means the non-trivial 2-tangle still exist in $p(L)$ and this means $p(L)$ is not the projection of a hyperbolic link and therefore not b-prime. This is a contradiction so L must be prime.

This concludes the proof. □

With Theorem 2 we are able to generate all b-prime FAL from the set of prime links. This is useful and can be done by hand quite quickly. The Figure 6 is an example of the process.

3 Future work and open questions

A useful line of research would be finding the upper and lower bounds or the exact number of b-prime FAL with a given number of crossing arcs. A lower bound certainly is the number of prime links with a given number of crossings, however it is theorized that the number of b-prime FAL is much greater than the number of prime links. This is because non-minimal twists can generate different b-prime FAL. So one prime link can generate numerous b-prime FAL, however not all may be unique because of symmetry. This complicates the process of finding the exact number of b-prime FAL.

It would be a wonderful step to consider the previous work on FAL to see what more can be understood about FAL. The authors suggest considering the work for Morgan et. al[2] and Ransom[3]. The b-prime FAL could be searched to find which are algebraic. This theorem is simply the foundation for numerous lines of research.

4 Conclusion

This paper has discussed belted-sums and flype orbits. Then complete augmentation on a prime knot was defined. Then the main result of the paper was proven; that the complete augmentation of a prime link will produce a b-prime FAL, and reversing the operation on a b-prime FAL will return a prime link. Section 3 suggests where this work could be continued, with finding the bounds or exact number of b-prime FAL with a given number of crossing circles.

References

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