Kernels of Algebraic Curvature Tensors of Mixed Symmetric and Skew-Symmetric Build

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Abstract

The kernel of an algebraic curvature tensor is a fundamental subspace that can be used to distinguish between different algebraic curvature tensors. Kernels of algebraic curvature tensors built only of canonical algebraic curvature tensors of a single build (symmetric or skew-symmetric) were investigated in [8] and [2], respectively. We consider the kernel of an algebraic curvature tensor R built from canonical algebraic curvature tensors of *both* symmetric and skew-symmetric build. An obvious way to ensure that the kernel of R is nontrivial is to choose the involved bilinear forms such that the intersection of their kernels is nontrivial. We present a construction wherein this intersection is trivial but the kernel of R is nontrivial. We also show how many bilinear forms satisfying certain conditions are needed in order for R to have a kernel of any allowable dimension.

1 Introduction

Throughout, let V be a finite-dimensional vector space over $\mathbb{R}$.

Definition 1.0.1. *An* ***algebraic curvature tensor*** *is a multilinear function* $R : V^4 \to \mathbb{R}$ *such that:*

a) $R(x, y, z, w) = -R(y, x, z, w)$*,*

b) $R(x, y, z, w) = R(z, w, x, y)$*, and*

c) $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

for all $x, y, z, w \in V$.

The set of all algebraic curvature tensors on V is denoted $\mathcal{A}(V)$. An algebraic curvature tensor is meant to mimic the algebraic properties of the Riemann curvature tensor at a given point of a manifold. An important subspace is the kernel of an algebraic curvature tensor:

Definition 1.0.2. *Let $R \in \mathcal{A}(V)$. The **kernel** of R is*

$$ker(R) := \{x \in V \mid R(x, y, z, w) = 0 \text{ for all } y, z, w \in V\}.$$

The defining properties of algebraic curvature tensors can be used to show (see [3]) that the definition of the kernel of an algebraic curvature tensor is not biased towards the first entry, that is,

$$\begin{aligned} ker(R) &= \{y \in V \mid R(x, y, z, w) = 0 \text{ for all } x, z, w \in V\} \\ &= \{z \in V \mid R(x, y, z, w) = 0 \text{ for all } x, y, w \in V\} \\ &= \{w \in V \mid R(x, y, z, w) = 0 \text{ for all } x, y, z \in V\}. \end{aligned}$$

Kernels are worthy of study because they are one way to distinguish between algebraic curvature tensors. The goal of this paper will be to demonstrate when an algebraic curvature tensor of a specific build has nontrivial kernel, if $dim(V) \geq 3$. The following proposition illustrates why we do not consider $dim(V) = 2$:

Proposition. *[8] If $dim(V) = n$ and $R \in \mathcal{A}(V)$, then $dim(ker(R)) \neq (n-1)$.*

Proof. (Included for convenience): First, recall from linear algebra that if $W \subseteq V$ is a subspace and $\{e_1, \ldots, e_k\}$ is a basis for W, then we can extend to a basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_n\}$ of V. Assume, towards a contradiction, that $dim(ker(R)) = n - 1$. Then there exists a basis $\{e_1, \ldots, e_{n-1}\}$ of $ker(R)$ that can be extended to a basis $\{e_1, \ldots, e_{n-1}, e_n\}$ of V. Thus $V = ker(R) \oplus span\{e_n\}$, so for $v \in V$ we can write $v = k_v + \lambda_v e_n$, with $k_v \in ker(R)$ and $\lambda_v \in \mathbb{R}$. Then for $x, y, z, w \in V$, we write

$$\begin{aligned} x &= k_x + \lambda_x e_n, \\ y &= k_y + \lambda_y e_n, \\ z &= k_z + \lambda_z e_n, \\ w &= k_w + \lambda_w e_n \end{aligned}$$

and use the fact that $k_x, k_y, k_z, k_w \in ker(R)$ to conclude that

$$R(x, y, z, w) = \lambda_x \lambda_y \lambda_z \lambda_w R(e_n, e_n, e_n, e_n) = 0.$$

This means $R = 0$ and therefore $dim(ker(R)) = dim(V) = n$, which is a contradiction because we assumed $dim(ker(R)) = n - 1$. Thus it cannot be true that $dim(ker(R)) = n - 1$. □

Therefore, the zero tensor is the only algebraic curvature tensor that has nontrivial kernel if $dim(V) = 2$. Given a symmetric bilinear form φ on V, the **canonical algebraic curvature tensor** R_φ associated to φ is

$$R_\varphi(x, y, z, w) := \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

Given a skew-symmetric bilinear form ψ on V, the canonical algebraic curvature tensor R_ψ associated to ψ is

$$R_\psi(x, y, z, w) := \psi(x, w)\psi(y, z) - \psi(x, z)\psi(y, w) - 2\psi(x, y)\psi(z, w).$$

It will be clear from context if a given canonical algebraic curvature tensor is of symmetric or skew-symmetric build. It is easy to see that $R_{a\tau} = a^2 R_\tau$ for all $a \in \mathbb{R}$ and any bilinear form τ on V. In particular, $R_{-\tau} = R_\tau$. It is known [5] that the sets of canonical algebraic curvature tensors of either symmetric or skew-symmetric build are both spanning sets of $\mathcal{A}(V)$:

$$span\{R_\varphi \mid \varphi \in S^2(V^*)\} = span\{R_\psi \mid \psi \in \Lambda^2(V^*)\} = \mathcal{A}(V).$$

Gilkey has shown [6] that $ker(R_\psi) = ker(\psi)$ and $ker(R_\varphi) = ker(\varphi)$ if $rank(\varphi) \neq 1$. We now define a few key terms.

Definition 1.0.3. *Let τ be any bilinear form on V.*

*a) τ is **positive definite** if $\tau(x, x) > 0$ for all nonzero $x \in V$.*

*b) τ is **nondegenerate** if $\tau(x, y) = 0$ for all $y \in V$ implies $x = 0$.*

Throughout, let φ and φ_i denote symmetric bilinear forms on V, let ψ and ψ_i denote skew-symemtric bilinear forms on V, and let $\varepsilon, \varepsilon_i \in \{-1, 1\}$.

Several previous results about kernels of algebraic curvature tensors guided this study. Kernels of algebraic curvature tensors built only of canonical algebraic curvature tensors with a symmetric build were studied by Strieby [8]:

Theorem. *[8] Let $dim(V) = n \geq 3$. Let φ_1 be positive definite. Then $dim(ker(R_{\varphi_1} + \varepsilon R_{\varphi_2})) \in \{0, 1, n\}$.*

Kernels of algebraic curvature tensors built only of canonical algebraic curvature tensors with a skew-symmetric build were studied by Brundan [2]. Note the choice of signs in the following two results.

Theorem. *[2] Let $R := \sum_{i=1}^{k} R_{\psi_i}$. Then $ker(R) = \bigcap_{i=1}^{k} ker(R_{\psi_i})$.*

Theorem. *[2] Let $R := R_{\psi_1} - R_{\psi_2}$. Then either $ker(R) = ker(R_{\psi_1}) \cap ker(R_{\psi_2})$ or $\psi_1 = \pm\psi_2$, in which case $R = 0$ and $ker(R) = V$.*

The above results have a certain rigidity: the kernel of an algebraic curvature tensor built only of canonical algebraic curvature tensors of symmetric build cannot be of *any* allowable dimension, while the kernel of an algebraic curvature tensor built only of canonical algebraic curvature tensors of skew-symmetric build is no bigger than the kernel of the individual bilinear forms involved. The goal of this work will be to expand on these results by considering algebraic curvature tensors built from canonical algebraic curvature tensors of *both* symmetric and skew-symmetric build. We consider algebraic curvature tensors of the form

$$R := R_{\varphi} + \sum_{i=1}^{k} \varepsilon_i R_{\psi_i}.$$

We can assume that the only coefficients in the above linear combination are 1 and -1, because $R_{a\psi} = a^2 R_{\psi}$. We can assume that the coefficient on R_{φ} is equal to 1, because $ker(R) = ker(-R)$ for any $R \in \mathcal{A}(V)$.
We aim to show what conditions on φ and the ψ_i's are sufficient in order for R to have nontrivial kernel. Since R_{φ} is the zero tensor if the rank of φ is 0 or 1 [6], we only consider the case when $rank(\varphi) \geq 2$. An easy way to ensure that R has nontrivial kernel is to choose φ and the ψ_j's such that

$$K := ker(\varphi) \bigcap_{i=1}^{k} ker(\psi_i) \neq \{0\},$$

which implies that $K \subseteq ker(R)$ and $ker(R) \neq \{0\}$. As shown in [7], any $R \in \mathcal{A}(V)$ can be written as a linear combination of canonical algebraic curvature tensors built from bilinear forms of full rank. Therefore, there must exist a case where $K = \{0\}$ but $ker(R) \neq 0$. A reasonable first step toward finding such a construction is to set φ to be positive definite. This ensures that $K = \{0\}$, and that there exists a basis $\mathcal{B}$ of V such that $\mathcal{B}$ is orthonormal with respect to φ and any skew-symmetric bilinear form ψ is block-diagonal with respect to $\mathcal{B}$ [1].

We now present a method we use in this work to find $ker(R)$. Let $\alpha \in ker(R)$, let $\{e_1, \ldots, e_n\}$ be a basis for V, and write $\alpha = \gamma_1 e_1 + \cdots + \gamma_n e_n$ for $\gamma_i \in \mathbb{R}$. We can solve for the γ_i's by considering the system of equations obtained from all nonzero curvature entries, up to symmetry, of the form $R(\alpha, e_i, e_j, e_k)$, where $i, j, k \in \{1, \ldots, n\}$. Note that repeating α in the inputs to R is not necessary, as doing so only produces a linear combination of already existing equations: for any $e_i, e_j \in \mathcal{B}$, note that

$$\begin{aligned} R(\alpha, \alpha, e_i, e_j) &= R(e_i, e_j, \alpha, \alpha) = 0, \text{ and} \\ -R(\alpha, e_i, \alpha, e_j) &= R(\alpha, e_i, e_j, \alpha) = R(\alpha, e_i, e_j, \gamma_1 e_1 + \cdots + \gamma_n e_n) \\ &= \gamma_1 R(\alpha, e_i, e_j, e_1) + \cdots + \gamma_n R(\alpha, e_i, e_j, e_n). \end{aligned}$$

In Section 2, we investigate the kernel of an algebraic curvature tensor built of just two canonical algebraic curvature tensors: one of symmetric and one of skew-symmetric build. In Section 3, we show how to construct an algebraic curvature tensor that has a kernel of any allowable dimension. Our construction does not require φ to be positive definite but still ensures $K = \{0\}$. We also present a lower bound on how many bilinear forms satisfying certain conditions are needed to ensure that $ker(R) \neq \{0\}$. Finally, in Section 4, we present future directions of study.

2 Kernel of $R_\varphi + \varepsilon R_\psi$

In this section, we compute the kernel of an algebraic curvature tensor built of one canonical algebraic curvature tensor of symmetric build and one canonical algebraic curvature tensor of skew-symmetric build. Let $R := R_\varphi + \varepsilon R_\psi$. In Section 2.1, we show that if φ is positive definite, then $ker(R) = \{0\}$. In Section 2.2, we consider only the case when $dim(V) = 3$ and show that if φ is not necessarily positive definite and $rank(\varphi) \geq 2$, then $ker(R)$ is equal to the intersection of the kernels of the bilinear forms used to construct R.

2.1 Positive definite φ

Our goal is to show that the kernel of R is trivial if φ is positive definite and $dim(V) \geq 3$. If φ is positive definite, there is a basis

$$\mathcal{B} = \{e_1, f_1, \ldots, e_s, f_s, x_1, \ldots, x_t\}$$

of V such that $2s + t = n$, $\varphi = I_n$, and ψ is block-diagonal with respect to $\mathcal{B}$ [1]. Concretely, $\psi(e_i, f_i) = b_i > 0$ for $i \in \{1, \ldots, s\}$ and $span\{x_i\} = ker(\psi)$.

Lemma 2.1.1. *If φ is positive definite, then $R_\varphi(e_i, e_j, e_j, e_i) = R_\varphi(f_i, f_j, f_j, f_i) = R_\varphi(x_i, x_j, x_j, x_i) = 1$ for $i \neq j$.*

Proof. Let $i \neq j$. Note that

$$R_\varphi(e_i, e_j, e_j, e_i) = \varphi(e_i, e_i)\varphi(e_j, e_j) - \varphi(e_i, e_j)\varphi(e_j, e_i) = 1.$$

As $\varphi = I_n$, it is also the case that $R_\varphi(f_i, f_j, f_j, f_i) = R_\varphi(x_i, x_j, x_j, x_i) = 1$. □

Lemma 2.1.2. *If ψ is skew-symmetric, then $R_\psi(e_i, e_j, e_j, e_i) = R_\psi(f_i, f_j, f_j, f_i) = R_\psi(x_i, x_j, x_j, x_i) = 0$ for $i \neq j$.*

Proof. Let $i, j \in \{1, \ldots, s\}$ and $i \neq j$. Note

$$R_\psi(e_i, e_j, e_j, e_i) = \psi(e_i, e_i)\psi(e_j, e_j) - \psi(e_i, e_j)\psi(e_j, e_i) - 2\psi(e_i, e_j)\psi(e_j, e_i) = 0.$$

As the only nonzero entries of ψ are those of the form $\psi(e_i, f_i)$, it is also the case that $R_\psi(f_i, f_j, f_j, f_i) = R_\psi(x_i, x_j, x_j, x_i) = 0$. □

We now show that if φ is positive definite, then $ker(R) = \{0\}$.

Theorem 2.1.3. *Let $dim(V) = n \geq 3$ and let φ be positive definite. Then $ker(R_\varphi + \varepsilon R_\psi) = \{0\}$.*

Proof. Let $\alpha \in ker(R)$. Note

$$\alpha = p_1 e_1 + q_1 f_1 + \cdots + p_s e_s + q_s f_s + r_1 x_1 + \cdots + r_t x_t$$

for $p_i, q_i, r_i \in \mathbb{R}$. We will show that $\alpha = 0$.
First, fix $i \in \{1, \ldots, s\}$ and let $j \in \{1, \ldots, s\}$ with $j \neq i$. Note that, by Lemma 2.1.1 and Lemma 2.1.2,

$$0 = R(\alpha, e_j, e_j, e_i) = \varphi(\alpha, e_i) = p_i \text{ and}$$
$$0 = R(\alpha, f_j, f_j, f_i) = \varphi(\alpha, f_i) = q_i.$$

Next, fix $i \in \{1, \ldots, t\}$ and let $j \in \{1, \ldots, t\}$ with $j \neq i$. By Lemma 2.1.1 and Lemma 2.1.2,

$$0 = R(\alpha, x_j, x_j, x_i) = \varphi(\alpha, x_i) = r_i.$$

Thus $p_i = q_i = 0$ for all $i \in \{1, \ldots, s\}$ and $r_i = 0$ for all $i \in \{1, \ldots, t\}$, so $\alpha = 0$ and $ker(R) = \{0\}$. □

2.2 Any φ; dim(V) = 3

We now consider the case when φ is not necessarily positive definite and show that if $dim(V) = 3$, then the kernel of R is equal to the intersection of the kernels of the bilinear forms that make up R.

Theorem 2.2.1. *Let $dim(V) = 3$. Let φ be any symmetric bilinear form on V of rank 2 or higher, and assume φ is not necessarily positive definite. Then $ker(R) = ker(\varphi) \cap ker(\psi)$.*

Proof. Let $dim(V) = 3$. As φ is symmetric, there exists a basis $\mathcal{B} = \{e_1, e_2, e_3\}$ of V such that φ is diagonal with respect to $\mathcal{B}$ and only takes values of 0, 1, or -1 on the diagonal. As $rank(\varphi) \geq 2$, we can reorder the basis vectors so that the only zero diagonal entry of φ, if one exists, is $\varphi(e_3, e_3)$. As $R_\varphi = R_{-\varphi}$, we can also assume, without loss of generality, that $\varphi(e_1, e_1) = 1$. Therefore, let

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}, \quad \psi = \begin{bmatrix} 0 & y & z \\ -y & 0 & w \\ -z & -w & 0 \end{bmatrix}$$

for $\delta_2 \in \{-1, 1\}$ and $\delta_3 \in \{-1, 0, 1\}$. Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$. We consider all possible

$\{R(\alpha, e_i, e_j, e_k)\}$ and obtain

$$\gamma_2 yw + \gamma_3 zw = 0, \tag{2.1}$$
$$\gamma_1 yz - \gamma_3 zw = 0, \tag{2.2}$$
$$\gamma_1 yz + \gamma_2 yw = 0, \tag{2.3}$$
$$\gamma_2(\delta_2 + 3\varepsilon y^2) + 3\varepsilon\gamma_3 yz = 0, \tag{2.4}$$
$$\gamma_3(-\delta_3 - 3\varepsilon z^2) - 3\varepsilon\gamma_2 yz = 0, \tag{2.5}$$
$$\gamma_1(-\delta_2 - 3\varepsilon y^2) + 3\varepsilon\gamma_3 yw = 0, \tag{2.6}$$
$$\gamma_3(\delta_2\delta_3 + 3\varepsilon w^2) - 3\varepsilon\gamma_1 yw = 0, \tag{2.7}$$
$$\gamma_2(-\delta_2\delta_3 - 3\varepsilon w^2) - 3\varepsilon\gamma_1 zw = 0, \tag{2.8}$$
$$\gamma_1(-\delta_3 - 3\varepsilon z^2) - 3\varepsilon\gamma_2 zw = 0. \tag{2.9}$$

Case 1: Let $z, w \neq 0$. Note that (2.2) gives

$$\gamma_1 y = \gamma_3 w \implies \gamma_1 yw = \gamma_3 w^2.$$

Substituting into (2.7) gives $\gamma_3\delta_3 = 0$. If $\delta_3 \neq 0$, then $ker(\varphi) \cap ker(\psi) = \{0\}$ and $\gamma_3 = 0$. Then (2.1) and (2.2) imply that either $y = 0$ or $\gamma_1 = \gamma_2 = 0$. If $y = 0$, then (2.4) gives $\gamma_2 = 0$. Then (2.8) gives $\gamma_1 = 0$, and the kernel of R is trivial; in fact, $ker(R) = ker(\varphi) \cap ker(\psi)$.
If $\delta_3 = 0$, then $ker(\varphi) = span\{e_3\}$ and $ker(\varphi) \cap ker(\psi) = \{0\}$, as $e_3 \in ker(\psi)$ only if $z = w = 0$. Note (2.2) gives $\gamma_3 = \frac{y}{w}\gamma_1$, and (2.3) gives $\gamma_2 = -\frac{z}{w}\gamma_1$. Substituting these values into (2.4) and (2.6) gives $\gamma_1 = 0 \implies \gamma_1 = \gamma_2 = \gamma_3 = 0 \implies ker(R) = \{0\} = ker(\varphi) \cap ker(\psi)$.

Case 2: Let $z = 0$. Note that (2.5) and (2.9) imply that either $\delta_3 = 0$ or $\gamma_1 = \gamma_3 = 0$. If $\delta_3 \neq 0$, then $ker(\varphi) = \{0\}$ and $\gamma_1 = \gamma_3 = 0$. If $\gamma_2 \neq 0$, say $\gamma_2 = 1$, then (2.4) implies $y \neq 0$; thus (2.1) implies $w = 0$. Then (2.8) implies that $\delta_2 = 0$ or $\delta_3 = 0$, which is a contradiction. Thus $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and $ker(R) = \{0\} = ker(\varphi) \cap ker(\psi)$.
If $\delta_3 = 0$, then $ker(\varphi) = span\{e_3\}$ and (2.8) implies that either $\gamma_2 = 0$ or $w = 0$. If $w = 0$, then $ker(\psi) = span\{e_3\}$. We are left with the system of equations

$$\gamma_2(\delta_2 + 3\varepsilon y^2) = 0$$
$$\gamma_1(-\delta_2 - 3\varepsilon y^2) = 0,$$

which implies $span\{e_3\} \subseteq ker(R)$. As $dim(ker(R)) \neq 2$, it must be true that $ker(R) = span\{e_3\}$. Then $ker(R) = ker(\varphi) \cap ker(\psi)$.

Now let $\gamma_2 = 0$. We are left with the system of equations

$$\gamma_1(-\delta_2 - 3\varepsilon y^2) + 3\varepsilon\gamma_3 yw = 0$$
$$-\gamma_1 yw + \gamma_3 w^2 = 0,$$

which has a nonzero solution if $w = 0$. Then $e_3 \in ker(R)$ and, as above, $ker(R) = span\{e_3\} = ker(\varphi) \cap ker(\psi)$.

Case 3: Let $w = 0$. Then (2.7) and (2.8) imply that either $\delta_3 = 0$ or $\gamma_2 = \gamma_3 = 0$. If $\delta_3 = 0$ then $ker(\varphi) = span\{e_3\}$, and (2.9) implies that either $\gamma_1 = 0$ or $z = 0$. If $z = 0$, then $ker(\psi) = span\{e_3\}$ and $ker(\varphi) \cap ker(\psi) = span\{e_3\}$. We are left with the system

$$\gamma_2(\delta_2 + 3\varepsilon y^2) = 0$$
$$\gamma_1(-\delta_2 - 3\varepsilon y^2) = 0,$$

which means $e_3 \in ker(R)$ and therefore, as above, $ker(R) = span\{e_3\}$. Then $ker(R) = ker(\varphi) \cap ker(\psi)$.
Now let $\gamma_1 = 0$. We are left with the system of equations

$$\gamma_2(\delta_2 + 3\varepsilon y^2) + 3\varepsilon\gamma_3 yz = 0$$
$$\gamma_2 yz + \gamma_3 z^2 = 0,$$

which has a nonzero solution if $z = 0$. Then $e_3 \in ker(R)$ and, as before, $ker(R) = span\{e_3\} = ker(\varphi) \cap ker(\psi)$.
If $\gamma_2 = \gamma_3 = 0$ and $\delta_3 \neq 0$, then $ker(\varphi) = \{0\}$. If $\gamma_1 \neq 0$, then (2.2) implies that $y = 0$ or $z = 0$. If $y = 0$, then (2.6) gives the contradiction $\delta_2 = 0$. If $z = 0$, then (2.5) gives the contradiction $\delta_3 = 0$. Therefore, $\gamma_1 = 0$ and $ker(R) = \{0\} = ker(\varphi) \cap ker(\psi)$. □

Note that this result is similar in rigidity to the previously discussed results of [2]: the kernel of R is no bigger than the kernel of either of the bilinear forms involved in the construction of R.

3 In search of a nontrivial kernel

The goal of this section is to present a construction in which an algebraic curvature tensor of the form

$$R := R_\varphi + \sum_{i=1}^{k} \varepsilon_i R_{\psi_i}$$

has a kernel of any allowable dimension, despite the fact that the intersection of the kernels of the bilinear forms used to build R is trivial. In Section 3.1, we consider the case when $dim(V) = 3$ and φ is positive definite. This leads us to a construction that is valid in any finite dimension and does not require that φ be positive definite. We present this construction in Section 3.2, where we also give a lower bound on how many bilinear forms satisfying certain conditions are required for R to have a kernel of a given dimension.

3.1 Positive definite φ, $\dim(V) = 3$

Section 2 shows that if φ is positive definite, one symmetric and one skew-symmetric bilinear form are not enough to build an algebraic curvature tensor with nontrivial kernel. We now show when it is possible for an algebraic curvature tensor built from one symmetric positive definite bilinear form and two skew-symmetric bilinear forms to have nontrivial kernel.

Let $dim(V) = 3$, let φ be positive definite, and set $R := R_\varphi + \varepsilon_1 R_{\psi_1} + \varepsilon_2 R_{\psi_2}$. Then there exists a basis $\mathcal{B}$ of V such that

$$\varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \psi_1 = \begin{bmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 0 & y & z \\ -y & 0 & w \\ -z & -w & 0 \end{bmatrix}.$$

Lemma 3.1.1. *Unless $\varepsilon_1 = \varepsilon_2 = -1$, $ker(R) = \{0\}$.*

Proof. Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$ for $\gamma_i \in \mathbb{R}$. By considering all possible $\{R(\alpha, e_i, e_j, e_k)\}$, we obtain the following system of equations:

$$\gamma_2 yw + \gamma_3 zw = 0 \tag{3.1}$$
$$-\gamma_1 yz + \gamma_3 zw = 0 \tag{3.2}$$
$$\gamma_1 yz + \gamma_2 yw = 0 \tag{3.3}$$
$$\gamma_2(3\varepsilon_1 b^2 + 3\varepsilon_2 y^2 + 1) + 3\varepsilon_2\gamma_3 yz = 0 \tag{3.4}$$
$$\gamma_3(-3\varepsilon_2 z^2 - 1) - 3\varepsilon_2\gamma_2 yz = 0 \tag{3.5}$$
$$\gamma_1(-3\varepsilon_2 y^2 - 3\varepsilon_1 b^2 - 1) + 3\varepsilon_2\gamma_3 yw = 0 \tag{3.6}$$
$$\gamma_3(1 + 3\varepsilon_2 w^2) - 3\varepsilon_2\gamma_1 yw = 0 \tag{3.7}$$
$$\gamma_2(-3\varepsilon_2 w^2 - 1) - 3\varepsilon_2\gamma_1 zw = 0 \tag{3.8}$$
$$\gamma_1(-3\varepsilon_2 z^2 - 1) - 3\varepsilon_2\gamma_2 zw = 0. \tag{3.9}$$

First, let $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$ and assume, towards a contradiction, that one of the γ_i's is nonzero and that one of y, z, w is nonzero.

Case 1: Assume $\gamma_1 \neq 0$. Without loss of generality, let $\gamma_1 = 1$: if $\gamma_1 \neq 1$, simply scale α so that $\gamma_1 = 1$. Assume $z \neq 0$. Note (3.2) gives $y = \gamma_3 w$. Substituting into (3.6) gives $-3b^2 - 1 = 0$, which is a contradiction as $b \in \mathbb{R}$. Thus $z = 0$. Now, (3.9) gives the contradiction $-1 = 0$. Thus $\gamma_1 = 0$.

Case 2: Knowing that $\gamma_1 = 0$, let $\gamma_2 = 1$. Assume $w \neq 0$. Note (3.1) gives $y = -\gamma_3 z$. Then (3.4) gives the

contradiction $3b^2 + 1 = 0$, so $w = 0$. Now, (3.8) gives $-1 = 0$, so we must have $\gamma_2 = 0$.
Case 3: Knowing that $\gamma_1 = \gamma_2 = 0$, let $\gamma_3 = 1$. Note (3.5) gives $z = \sqrt{\frac{1}{3}}$ and (3.7) gives $w = \sqrt{\frac{1}{3}}$. However, we must have that either $z = 0$ or $w = 0$, by (3.2). Therefore $\gamma_3 = 0$.
We have shown that $ker(R) = \{0\}$ if $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. If $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, simply block-diagonalize ψ_2 with respect to φ. Then the above proof gives $ker(R) = \{0\}$.

Now, let $\varepsilon_1 = \varepsilon_2 = 1$. Again, towards a contradiction, assume that one of the γ_i's is nonzero and that one of y, z, w is nonzero.
Case 1: Assume $\gamma_1 = 1$ and let $y \neq 0$. Then (3.3) gives $z = -\gamma_2 w$. Substituting into (3.5) gives $-1 = 0$, so it must be true that $y = 0$. Then (3.6) yields a contradiction as $b \in \mathbb{R}$, so $\gamma_1 = 0$.
Case 2: Knowing that $\gamma_1 = 0$, let $\gamma_2 = 1$. By (3.3), either $y = 0$ or $w = 0$. If $y = 0$, then (3.4) gives a contradiction as $b \in \mathbb{R}$. If $w = 0$, then (3.8) yields the contradiction $-1 = 0$. Thus $\gamma_2 = 0$.
Case 3: Knowing that $\gamma_1 = \gamma_2 = 0$, let $\gamma_3 = 1$. Then (3.1) gives that either $z = 0$ or $w = 0$. If $z = 0$, then (3.5) yields the contradiction $-1 = 0$, and if $w = 0$, then (3.7) yields the contradiction $1 = 0$. Thus $\gamma_3 = 0$ and $ker(R) = \{0\}$ if $\varepsilon_1 = \varepsilon_2 = 1$.

□

Lemma 3.1.2. *Consider the same construct as in Lemma 3.1.1. If* $b \neq \sqrt{\frac{1}{3}}$*, then* $ker(R) = \{0\}$.

Proof. By Lemma 3.1.1, we know that $ker(R) = \{0\}$ if it is not true that $\varepsilon_1 = \varepsilon_2 = -1$. We therefore set $\varepsilon_1 = \varepsilon_2 = -1$ and show that $ker(R) = \{0\}$ if $b \neq \sqrt{\frac{1}{3}}$. Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$. We consider the system of equations produced in the proof of Lemma 3.1.1.
Case 1: Assume $\gamma_1 = 1$. Then (3.2) gives $y = \gamma_3 w$, if $z \neq 0$. Substituting into (3.6) gives the contradiction $3b^2 = 1$. Thus $z = 0$. Then (3.9) gives the contradiction $1 = 0$, so $\gamma_1 = 0$.
Case 2: Assume $\gamma_2 = 1$. Then (3.3) implies that either $y = 0$ or $w = 0$. If $y = 0$, then (3.4) gives the contradiction $-3b^2 = -1$. If $w = 0$, then (3.8) gives the contradiction $-1 = 0$. Thus $\gamma_2 = 0$.
Case 3: Assume $\gamma_3 = 1$. Then (3.1) gives that either $z = 0$ or $w = 0$. If $z = 0$ then (3.5) gives the contradiction $-1 = 0$, and if $w = 0$ then (3.7) gives the contradiction $1 = 0$. Thus $\gamma_3 = 0$ and $ker(R) = \{0\}$ if $b \neq \sqrt{\frac{1}{3}}$. □

We present a necessary and sufficient condition for R to be able to have a one-dimensional kernel:

Theorem 3.1.3. *It is possible to choose* ψ_2 *so that* $dim(ker(R)) = 1$ *if and only if* $\varepsilon_1 = \varepsilon_2 = -1$ *and* $b = \sqrt{\frac{1}{3}}$.

Proof. First, assume that it is possible to choose ψ_2 so that $dim(ker(R)) = 1$. Then Lemma 3.1.1 and Lemma 3.1.2 tell us that $\varepsilon_1 = \varepsilon_2 = -1$ and $b = \sqrt{\frac{1}{3}}$.
Now, let $\varepsilon_1 = \varepsilon_2 = -1$ and $b = \sqrt{\frac{1}{3}}$. We show that we can choose y, z, w such that $dim(ker(R)) = 1$. Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$ for $\gamma_i \in \mathbb{R}$. We again consider the system of equations used in the proof of Lemma 3.1.1. Set $y = w = 0$, $z = \sqrt{\frac{1}{3}}$. Note (3.7) gives $\gamma_3 = 0$ and (3.8) gives $\gamma_2 = 0$, while γ_1 can take any value. Thus $ker(R) = span\{e_1\}$ and $dim(ker(R)) = 1$. □

Remark 3.1.4. In the proof of Theorem 3.1.3, we could have set $y = z = 0$ and $w = \sqrt{\frac{1}{3}}$, in which case $ker(R) = span\{e_2\}$.

Proposition 3.1.5. *If R is as in Theorem 3.1.3, it is not possible that $ker(R) = span\{e_3\}$.*

Proof. Assume, towards a contradiction, that $\gamma_3 = 1$. Then (3.1) gives $z = -\gamma_2 y$ if $w \neq 0$. Substituting into (3.5) gives $-1 = 0$, so it must be true that $w = 0$. Substituting into (3.7) gives $1 = 0$, which is a contradiction. Thus $\gamma_3 = 0$ and $ker(R) \neq span\{e_3\}$. □

Theorem 3.1.3 implies that if $dim(V) = 3$, we need one positive definite symmetric and two skew-symmetric bilinear forms of a particular construction in order for a nontrivial kernel to be possible: we must either have

$$\psi_1 = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \psi_2 = \begin{bmatrix} 0 & 0 & \sqrt{\frac{1}{3}} \\ 0 & 0 & 0 \\ -\sqrt{\frac{1}{3}} & 0 & 0 \end{bmatrix}, \text{ or}$$

$$\psi_1 = \begin{bmatrix} 0 & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{3}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \psi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{3}} \\ 0 & -\sqrt{\frac{1}{3}} & 0 \end{bmatrix}.$$

In both of the above cases, ψ_1 and ψ_2 have exactly two nonzero entries. This suggests a construction that allows for a nontrivial kernel and does not rely on φ being positive definite. Before exploring this construction in arbitrary dimension, we introduce some notation. Let $i < j$ and let ψ_{ij} denote a skew-symmetric bilinear form such that

a) $\psi_{ij}(e_i, e_j) \neq 0$ and

b) $\psi_{ij}(e_\ell, e_k) = 0$ if $(\ell, k) \neq (i, j)$.

Then if $dim(V) = 3$, and given our previous choice of basis, we either need

$$\psi_1 = \psi_{12} \text{ and } \psi_2 = \psi_{13} \quad \text{or} \quad \psi_1 = \psi_{12} \text{ and } \psi_2 = \psi_{23}$$

in order for it to be possible for the dimension of the kernel of R to be 1.

3.2 Nondegenerate φ, $\dim(V) = n \geq 3$

Recall that a goal of this paper was to construct an algebraic curvature tensor with a kernel of any allowable dimension without intersecting the kernels of the bilinear forms involved. To this end, we investigate the construction from Section 3.1 in arbitrary dimension and without assuming φ is positive definite. Let $dim(V) =$

$n \geq 3$ and let φ be nondegenerate. Note that there exists a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of V such that

$$\varphi = \begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix}$$

with respect to $\mathcal{B}$ [4]. Consider an indexing set $S \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $i < j$ for all $(i, j) \in S$. Consider only the construction from Section 3.1, that is, let

$$R := R_\varphi + \sum_{(i,j) \in S} \varepsilon_{ij} R_{\psi_{ij}}.$$

Note that $ker(\varphi) = \{0\}$, so the intersection of the kernels of the bilinear forms involved in the construction of R is trivial. At the risk of being informal, and to present our results in the most intuitive way, we introduce two new terms.

Definition 3.2.1. *Let e_i, $e_j \in \mathcal{B}$ for $i < j$. We say e_i is* ***friends with*** *e_j if $(i, j) \in S$.*

Note that e_i is friends with e_j if one of the canonical algebraic curvature tensors out of which R is built is $R_{\psi_{ij}}$.

Definition 3.2.2. *Let $e_i \in \mathcal{B}$. We say e_i is* ***popular*** *if e_i is friends with every $e_j \in \mathcal{B}$ for $j \neq i$.*

Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \cdots + \gamma_n e_n$ for $\gamma_i \in \mathbb{R}$. Before showing when R must have a trivial kernel, we present a useful lemma:

Lemma 3.2.3. *If e_i is not popular, then $\gamma_i = 0$.*

Proof. There exists an $e_j \in \mathcal{B}$ that is not friends with e_i. Then

$$R(\alpha, e_j, e_j, e_i) = \pm\gamma_i = 0.$$

□

Theorem 3.2.4. *If no basis vector is popular, then $ker(R) = \{0\}$.*

Proof. By Lemma 3.2.3, $\gamma_i = 0$ for all i. □

We now state our main result, which gives a way to ensure that R has a kernel of any allowable dimension.

Theorem 3.2.5. *Let $m \in \{1, \ldots, n-2\} \cup \{n\}$. Let $\delta_{ij} := \varphi(e_i, e_i)\varphi(e_j, e_j)$. Then $dim(ker(R)) = m$ if and only if* *a) At least m basis vectors are popular, and*

b) $\delta_{i_k j} + 3\varepsilon_{i_k j} a_{i_k j}^2 = 0$ for exactly m indexes $\{i_1, \ldots, i_m\}$.

Proof. First, we assume $dim(ker(R)) = m$ and show that $a)$ and $b)$ must hold. Assume, towards a contradiction, that less than m basis vectors are popular. Note that without loss of generality, we can assume that $\{e_1, \ldots, e_\ell\}$ are popular for $\ell < m$. If not, simply relabel S so that the first ℓ basis vectors are popular. By Lemma 3.2.3, $\gamma_{\ell+1} = \cdots = \gamma_m = \cdots = \gamma_n = 0$. The number of nonzero γ_i's is therefore at most ℓ and $dim(ker(R)) \leq \ell < m$, which is a contradiction. Next, let $\ell < m$ and assume $\delta_{i_k j} + 3\varepsilon_{i_k j} a_{i_k j}^2 = 0$ for only ℓ indexes $\{i_1, \ldots, i_\ell\}$. Note that $\delta_{i_k j} + 3\varepsilon_{i_k j} a_{i_k j}^2 \neq 0$ implies $\gamma_{i_k} = 0$. Therefore, the number of γ_i's that are equal to 0 is greater than m, so $dim(ker(R)) \neq m$. Now, let $p > m$ and assume $\delta_{i_k j} + 3\varepsilon_{i_k j} a_{i_k j}^2 = 0$ for p indexes $\{i_1, \ldots, i_p\}$. Then the number of γ_i's that are equal to 0 is less than m, so $dim(ker(R)) \neq m$. Therefore, if $dim(ker(R)) = m$ then $a)$ and $b)$ must hold.

Next, we assume $a)$ and $b)$ both hold and show that $dim(ker(R)) = m$. Let $\alpha \in ker(R)$ and write $\alpha = \gamma_1 e_1 + \cdots + \gamma_n e_n$ for $\gamma_i \in \mathbb{R}$. Note that the only potentially nonzero curvature entries will be those of the form $R(\alpha, e_i, e_i, e_j)$ and $R(\alpha, e_j, e_j, e_i)$ for $(i, j) \in S$. Without loss of generality, assume that $\{e_1, \ldots, e_k\}$ are popular for $k > m$. We can further relabel S to assume, without loss of generality, that $\delta_{i_k j} + 3\varepsilon_{i_k j} a_{i_k j}^2 = 0$ for only

$i_k \in \{1, \ldots, m\}$. We obtain the following system of equations:

$$\gamma_1(\delta_{12} + 3\varepsilon_{12}a_{12}^2) = 0,$$
$$\gamma_2(\delta_{12} + 3\varepsilon_{12}a_{12}^2) = 0,$$
$$\vdots$$
$$\gamma_1(\delta_{1m} + 3\varepsilon_{1m}a_{1m}^2) = 0,$$
$$\gamma_m(\delta_{1m} + 3\varepsilon_{1m}a_{1m}^2) = 0,$$
$$\vdots$$
$$\gamma_1(\delta_{1k} + 3\varepsilon_{1k}a_{1k}^2) = 0,$$
$$\gamma_k(\delta_{1k} + 3\varepsilon_{1k}a_{1k}^2) = 0,$$
$$\vdots$$
$$\gamma_1(\delta_{1n} + 3\varepsilon_{1n}a_{1n}^2) = 0,$$
$$\vdots$$
$$\gamma_2(\delta_{2n} + 3\varepsilon_{2n}a_{2n}^2) = 0,$$
$$\vdots$$
$$\gamma_m(\delta_{mn} + 3\varepsilon_{mn}a_{mn}^2) = 0,$$
$$\vdots$$
$$\gamma_k(\delta_{kn} + 3\varepsilon_{kn}a_{kn}^2) = 0.$$

Note that only $\gamma_1, \ldots, \gamma_m$ are free, so $ker(R) = span\{e_1, \ldots, e_m\}$ and $dim(ker(R)) = m$. □

We now give a few examples to illustrate the use of Theorem 3.2.5.

Example 3.2.6. Let $dim(V) = 4$ and let $S = \{(1,2), (1,3), (1,4), (2,3), (2,4)\}$. If φ is positive definite, then

$$R_\varphi - \frac{1}{3} \sum_{(i,j)\in S} R_{\psi_{ij}}$$

has a two-dimensional kernel, since only e_1 and e_2 are popular and therefore $ker(R) = span\{e_1, e_2\}$.

Example 3.2.7. Let $dim(V) = n$, let φ be positive definite, and assume all n basis vectors of V are popular. Then

$$R_\varphi = \frac{1}{3} \sum_{(i,j)\in S} R_{\psi_{ij}}.$$

Example 3.2.8. Let

$$\varphi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then

$$0 = R_\varphi + \frac{1}{3}\left(-R_{\psi_{12}} + R_{\psi_{13}} + R_{\psi_{14}} + R_{\psi_{23}} + R_{\psi_{24}} - R_{\psi_{34}}\right).$$

Before presenting some corollaries of Theorem 3.2.5, we make a note:

Remark 3.2.9. Assume $\{e_1, \ldots, e_m\}$ are popular. Then the set

$$\begin{array}{llllllll} \{ & (1,2), & & & & & & \\ & (1,3), & (2,3), & & & & & \\ & (1,4), & (2,4), & (3,4), & & & & \\ & (1,5), & (2,5), & (3,5), & (4,5), & & & \\ & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & (m,m+1) & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & (1,n), & (2,n), & (3,n), & (4,n) & \ldots & (m,n) & \} \end{array}$$

is contained in S, and $|S| \geq \sum_{k=1}^{m}(n-k)$.

Corollary 3.2.10. *To have $dim(ker(R)) = m$, we need one φ and $\sum_{k=1}^{m}(n-k)$ ψ_i's.*

Corollary 3.2.11. *To have $dim(ker(R)) = n$, we need one φ and $\binom{n}{2}$ ψ_i's.*

Note that the construction given in Theorem 3.2.5 is restricted to algebraic curvature tensors built of canonical algebraic curvature tensors coming from exactly one positive definite symmetric bilinear form and any number of skew-symmetric bilinear forms that have exactly two nonzero entries.

4 Future Work

- Investigate if $ker(R) = ker(\varphi) \cap ker(\psi)$ if φ is symmetric of rank 2 or higher, φ is not necessarily positive definite, ψ is skew-symmetric, and $dim(V) > 3$.
- In our construction in Section 3.2, investigate how $ker(R)$ changes when φ has a kernel.

- Find a construction that requires fewer bilinear forms than the construction presented in Section 3.2. Investigate by allowing each skew-symmetric bilinear form to have more than two nonzero entires.
- Find a construction involving more than one symmetric bilinear form that allows for a nontrivial kernel.

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