On the Construction of Canonical Algebraic Curvature Tensors Using a Chosen Rank

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CSU San Bernardino REU 2022

July 21, 2022

Abstract

The goal of this project is to detail new methods for manipulating canonical tensors, as well as to describe some of the benefits the new methods provide to the field. Multiple new spanning sets for tensors have been found, as well as a link between the upper bound of $\eta(n)$ and the actual value of $\mu(n)$.

Contents

1 Preliminary Details

This paper is a compilation of several results related to algebraic curvature tensors and their mathematical representations. First, the basic details of ACTs will be explained, and some notation will be provided. Then, several methods of manipulating ACTs will be described. Some were already known, and those will cite the paper from which they originate, but some are new. The methods will be divided into ones related to symmetric-build ACTs and ones related to antisymmetric-build ACTs, though there is also one generic method which does not fit in either category. After that, we will explore the consequences of these methods and what they let us prove. Finally, a list of further questions will be presented.

Throughout, V denotes a real vector space of finite dimension n. A multilinear function $R : V \times V \times V \times V \to \mathbb{R}$ is an *algebraic curvature tensor* (ACT) if $\forall x, y, z, w \in V, R$ satisfies

$$R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w), \text{and the Bianchi Identity,}$$
$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.$$

The space of all ACTs on V is denoted $\mathcal{A}(V)$, and $dim(\mathcal{A}(V)) = \frac{n^2(n^2-1)}{12}$ ([4]). Given a symmetric bilinear form φ or an antisymmetric bilinear form ψ, we define the *canonical ACT* R_φ or R_ψ to be

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w), \text{ or}$$
$$R_\psi(x,y,z,w) = \psi(x,w)\psi(y,z) - \psi(x,z)\psi(y,w) - 2\psi(x,y)\psi(z,w).$$

If a canonical ACT uses the φ definition, we call it a symmetric-build ACT. Likewise, if it uses the ψ definition, we call it an antisymmetric-build ACT.

We use a 2-dimensional array to keep track of the values of φ or ψ. Following the same procedure as [8], if we assume that we have an orthonormal basis and a positive definite inner product, our array will have the same values as the matrix associated with φ or ψ. Using that, we are allowed to use "rank" as a well-defined term which applies to our arrays - an array's rank is the rank of the equivalent matrix. Furthermore, in our study, the conullity of a canonical ACT is equal to the rank of the associated array.

All antisymmetric-build ACTs have even conullity. If an ACT has conullity 3 or higher, it is equal to another ACT iff both ACTs have the same value array or value arrays which differ only by their sign [4]. $\mathcal{A}(V)$ is spanned by the set of symmetric-build ACTs, and also separately spanned by the set of antisymmetric-build ACTs [3]. To state that more explicitly, any algebraic curvature ACT can be represented as a linear combination of symmetric-build canonical ACTs, or as a linear combination of antisymmetric-build canonical ACTs. Gilkey also introduced terminology useful for this [4]: For an ACT $R, \nu(R)$ is the minimal number of symmetric-build ACTs needed to represent R, $\eta(R)$ is the minimal number of antisymmetric-build ACTs needed to represent R, and $\mu(R)$ is the minimal number of canonical ACTs required without restrictions on build type. More generally, for $n \geq 2$, $\nu(n)$ is defined as follows: over the set of all ACTs R where $dim(R) = n$, what is the highest value of $\nu(R)$? $\eta(n)$ and $\mu(n)$ follow the same pattern. [4]

While there are some reasonably good estimates on $\nu(n)$ and $\eta(n)$ for small values of n, even at $n = 4$ it is not known for sure whether the current estimates ($\nu(4) \leq 6, \eta(4) \leq 11$) are sharp. Furthermore, it was not previously known whether $\mu(3)$ was equal to 1 or 2. The upper bound on $\nu(n)$ is known to be $\frac{n(n+1)}{2}$ ([2]). However, it is not known whether $\mu(n)$ is better than $\nu(n)$ for any n, and our best upper bound for $\eta(n)$ is $\frac{n^2(n^2-1)}{12} - \binom{n-1}{2}$ (proven in [6]), which is only barely smaller than the dimension of $\mathcal{A}(V)$.

In the search to find better values for these upper bounds, some have found new interesting sub-questions related to them. For example, [1] found a way to convert any antisymmetric-build ACT of a given conullity into three ACTs of a conullity 2 higher, and [7] found an analogous result for symmetric-build ACTs. [7] then showed that for any $k \leq dim(V)$, it was possible to span $\mathcal{A}(V)$ using strictly ACTs of conullity greater than or equal to k. This was given the terminology $\nu_k(n)$, along with the analogous $\eta_k(n)$ and $\mu_k(n)$. No upper bounds other than the dimension of $\mathcal{A}(V)$ are known for these new values.

An algebraic curvature ACT is defined wholly by its components, which are the non-zero outputs of the ACT when given an input consisting only of basis vectors. To give an example, the 1234 component of an R is equal to $R(e_1, e_2, e_3, e_4)$ for some selected basis. Due to the original definition of an algebraic curvature ACT, only three kinds of components can be nonzero, named by the indices of the basis vectors involved: $ijji$, $ijki$, and $ijkl$ components. We also only care about components which are linearly independent from each other, so if a particular $ijji$ component is marked, we ignore the $ijij$ component. If two ACTs have the same components on the same basis, then they are the same.

1.1 Notation

For the sake of readability, throughout this entire paper, all zeroes inside of a matrix will be written as '.' instead. This is because there will be a lot of zeroes, and they can cause the eye to glaze over the actual important patterns that may show up. We will use the following notation for some equations in the poster:

Definition 1.1.1. ***Row Removal.*** *Let A be a $k \times k$ matrix. Define $A \setminus i$ to be a $k \times k$ matrix with the same entries as A, except that any entry where either index is equal to i is replaced with zero. The notation $A \setminus \{i, j\}$ means that entries where indices are equal to either i or j are replaced with zero.*

Definition 1.1.2. ***Row Isolation.*** *Let A be a $k \times k$ matrix. Define $A \perp i$ to be a $k \times k$ matrix that has zeroes for all its entries, except that any entry where either index is equal to i is the same as the corresponding entry in A. The notation $A \perp \{i, j\}$ is the same, but for entries where indices are equal to either i or j.*

Definition 1.1.3. ***Antisymmetric Twists.*** *Let A be an antisymmetric matrix. A twist on A is defined as the action of flipping the sign of one value and its antisymmetric partner within the matrix.*

For example, the following two matrices are twists of each other.

$$\begin{bmatrix} . & a \\ -a & . \end{bmatrix} \begin{bmatrix} . & -a \\ a & . \end{bmatrix}.$$

In a Jordan Block Form matrix, a twist will flip the sign of $ijkl$ components which touch the twisted block, but will leave $ijji$ components untouched. For this paper, an antisymmetric matrix is called 'twisted' if it has negative components above the diagonal. This relies on the assumption that when converting a matrix into Jordan Block Form, it is possible to force the values above the diagonal to be positive. Fortunately, that is actually something that we can do, so the assumption holds.

2 Manipulation of Arrays

In this section are a collection of useful array relationships that have been found, several of which are based on the theme of converting ACTs of a certain build and conullity into ACTs of the same build but a lower conullity. I call this process "downscaling", and it is either a "clean" downscale where all of the outputs are of the same conullity, or a "crude" downscale where the outputs are of at least two different conullities.

Note that there will be many times when a matrix is presented and required to be of a certain rank. This does not also restrict the dimension of V - assume that there are an arbitrary number of rows and columns populated only by zeroes, which are simply not shown. None of the relationships in this paper strictly require a specific dimension.

2.1 Symmetric Manipulation Techniques

While talking about symmetric ACTs, there will be times when an example is presented, and those examples will mention φ. In those examples, φ has rank 4 and is equal to the following:

$$\begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & c & . \\ . & . & . & d \end{bmatrix}.$$

Note that this only applies within the examples. Everywhere else, φ is considered to be a generic matrix.

Lemma 2.1.1. ***Symmetric Upscaling.*** *In [7], it was shown that any symmetric ACT can be converted into two symmetric ACTs of conullity one higher.*

Within this paper, this lemma is not used to actually convert any matrices, so it is sufficient to just note that it is true for later analysis.

Theorem 2.1.2. ***Clean Symmetric Downscaling.*** *If $\varphi : V \to V$, $\varphi = \varphi^*$, $Rk\varphi = n > 2$ and φ is diagonalized, then the following equation is true:*

$$R_\varphi = \frac{1}{n-2} \sum_{i=1}^{n} R_{A \backslash i}.$$

Proof. A diagonalized matrix of rank n only has up to $\binom{n}{2}$ components, all of the form $ijji$. We create n symmetric matrices of rank $n-1$, each one based on A but zeroing out a single eigenvalue as described in Definition 1.1.1. The components of these new matrices will also be of the form $ijji$, as they are also diagonalized. Let us look more closely at one of the new matrices, ϕ_k. This matrix, when expressed as an ACT, will have at most $\binom{n-1}{2}$ nonzero components, because it is conullity $n-1$. It will have the same components as R_φ, except for those components which contain k as an index. For any one component, there are exactly two matrices which will have zero instead of that component's value in the original ACT. This is because in symmetric diagonalized ACTs, each component is the product of two eigenvalues. Since we are zeroing out each eigenvalue exactly once, each component becomes zero exactly twice. Similarly, since the only change we make is that zeroing-out, the product is unmodified in every case other than the two where it becomes zero. This is consistent across every component in the original tensor, and so our final sum is $n-2$ times too large. We simply divide that out again to restore the balance. □

Here is an example. Since n is four, we need to divide by 2, which we do by simply multiplying the other side of the equation to fit.

$$2R_\varphi = R_A + R_B + R_C + R_D$$

$$A = \begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & c & . \\ . & . & . & . \end{bmatrix} B = \begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & . & . \\ . & . & . & d \end{bmatrix}$$

$$C = \begin{bmatrix} a & . & . & . \\ . & . & . & . \\ . & . & c & . \\ . & . & . & d \end{bmatrix} D = \begin{bmatrix} . & . & . & . \\ . & b & . & . \\ . & . & c & . \\ . & . & . & d \end{bmatrix}.$$

Note: Most of the proofs in the rest of the paper will be a little less precise than this one was. However, most of the incredibly precise and careful manipulation of components is not exactly necessary. All necessary steps to guarantee that the proof works are shown, and the reader may verify for themselves that the same results are found when doing everything with perfect accuracy.

Theorem 2.1.3. ***Crude Symmetric Downscaling.*** *If $\varphi : V \to V$, $\varphi = \varphi^*$, $Rk\varphi = n > 2$ and φ is diagonalized, then the following equation is true:*

$$2R_\varphi = R_{\varphi\setminus\{n\}+\varphi\perp\{n-1\}} + R_{\varphi\setminus\{n-1\}+\varphi\perp\{n\}} + 2R_{\varphi\perp\{n-1,n\}}.$$

Remark: To be clear, that complicated tangle in the subscripts means "A matrix which is the same as A, except that one value is doubled and one is removed", and it is unfortunate that no more efficient notation exists.

Proof. The two complicated ACTs cover all but one of the components. The first one covers the components which contain $n-1$ as an index twice, the components which contain n zero times, and all other components once. The second ACT behaves similarly, but with the twice and zero indices swapped. Together, they cover every component twice, except for the one that contains both $n-1$ and n. That one is added back in using a conullity 2 ACT, and the entire combination is divided by two. □

Here is an example:

$$2R_\varphi = R_A + R_B + 2R_C$$

$$A = \begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & 2c & . \\ . & . & . & . \end{bmatrix} \qquad B = \begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & . & . \\ . & . & . & 2d \end{bmatrix} \qquad C = \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & c & . \\ . & . & . & d \end{bmatrix}.$$

Theorem 2.1.4. ***Symmetric Downscaling by Several Conullities.*** *If $\varphi : V \to V$, $\varphi = \varphi^*$, $Rk\varphi = k > 2$ and φ is diagonalized, then for any $m \leq k-2$, it is possible to represent R_φ as the sum of canonical ACTs of conullity $k-m$. While a generic formula does not exist yet, there is an upper bound on how many ACTs are required,*

$$num_{ACTs} \leq \binom{k}{m}.$$

Proof. The theorem makes two claims - first, that creating such a sum is always possible, and second, that there will never need to be more than $\binom{k}{m}$ ACTs in the sum. The first claim is trivial to prove: Since we can always cleanly downscale by one rank, we simply do so repeatedly until we reach the desired rank. The second claim is more tricky. Our tactic from the first claim does not work, as it quickly generates far too many matrices. Instead, we create a new strategy based on expanding our previous ideas. When downscaling by one, we created n matrices, each one with one index zeroed out. Let us instead think of it as $\binom{k}{1}$ matrices. If we downscale by two, we will need to zero out two indices in order to fit into the new conullity, and we will have $\binom{k}{2}$ matrices as that is the number of ways to zero out two indices without repeats. Essentially, what we are doing is guaranteeing that we'll hit every single component by hitting them all too many times. If you select any two ACTs from our generated list, there are at least k components for which one has an actual value while the other has zero. This lack of repetition forces the entire list of $ijji$ components to be filled out, since those are the only components our generated list can have as non-zero entries. As our original ACT only had $ijji$ components due to being diagonalized, it is now fully defined. $\square$

Here is an example, where $m = 2$. In general, when the result is a group of conullity 2 tensors, no coefficient is required. However, it is not known whether a coefficient is required when the output is something other than conullity 2.

$$R_\varphi = R_A + R_B + R_C + R_D + R_E + R_F$$

$$A = \begin{bmatrix} a & . & . & . \\ . & b & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} B = \begin{bmatrix} a & . & . & . \\ . & . & . & . \\ . & . & c & . \\ . & . & . & . \end{bmatrix} C = \begin{bmatrix} a & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & d \end{bmatrix}$$

$$D = \begin{bmatrix} . & . & . & . \\ . & b & . & . \\ . & . & c & . \\ . & . & . & . \end{bmatrix} E = \begin{bmatrix} . & . & . & . \\ . & b & . & . \\ . & . & . & . \\ . & . & . & d \end{bmatrix} F = \begin{bmatrix} . & . & . & . \\ . & . & . & . \\ . & . & c & . \\ . & . & . & d \end{bmatrix}.$$

As you can see, there are six ACTs, which is equal to $\binom{4}{2}$.

2.2 Antisymmetric Manipulation Techniques

During this section, there will be examples that refer to R_ψ. For the purposes of those examples, ψ is equal to

$$\begin{bmatrix} . & a & . & . & . & . & . & . \\ -a & . & . & . & . & . & . & . \\ . & . & . & b & . & . & . & . \\ . & . & -b & . & . & . & . & . \\ . & . & . & . & . & c & . & . \\ . & . & . & . & -c & . & . & . \\ . & . & . & . & . & . & . & d \\ . & . & . & . & . & . & -d & . \end{bmatrix}.$$

Theorem 2.2.1. ***Conullity 2 Conversion.*** *For any $R_\varphi \in \mathcal{A}(V)$ where conullity$(R_\varphi) = 2$, there exists R_ψ such that $R_\varphi = \pm R_\psi$, where ψ can be chosen as either symmetric or antisymmetric.*

Proof. For an ACT R of conullity 2, there exists at least one basis which makes the only nonzero component of R the 1221 component. Let us say that the value of R_{1221} is x. Let us first consider the case where we want a symmetric ψ. An example of a symmetric matrix which generates the desired component would be

$$\begin{bmatrix} \sqrt{|x|} & . \\ . & \sqrt{|x|} \end{bmatrix}.$$

Note that if x is negative, we are still happy. We only asked that they are equivalent up to their sign, so as long as the absolute values match, everything is fine.

Now let us consider the case where we want an antisymmetric ψ. This is a bit more troublesome due to the more complicated equation for antisymmetric build, but not incredibly so. One example of an antisymmetric matrix which generates our desired component is

$$\begin{bmatrix} \cdot & \sqrt{\frac{|x|}{3}} \\ -\sqrt{\frac{|x|}{3}} & \cdot \end{bmatrix}.$$

Since we have shown examples of both types of ψ, we are done. $\square$

Theorem 2.2.2. ***Crude Antisymmetric Downscaling.*** *If $\psi : V \to V$, $\psi = -\psi^*$, and $Rk\psi = 2n > 4$, then the following equation is true so long as ψ is reduced to Jordan Block Form (JBF):*

$$R_\psi = \frac{1}{n-2} \sum_{i=1}^{n} R_{\psi \setminus \{2i,2i-1\}} - R_{\psi \perp \{2i,2i-1\}}.$$

Proof. For a JBF-reduced matrix of rank $2n$, the corresponding ACT has two categories of components which are nonzero, $ijji$ and $ijkl$. The $ijkl$ components come in triplets, and always have two indices from one block, and two indices from another. The $ijji$ components always have all their indices originate from one block. We create a linear combination of n antisymmetric-build ACTs of conullity $2n-2$, each one's corresponding array based on ψ but zeroing out a single Jordan block as described in Definition 1.1.1. Each $ijkl$ triplet is 'hit' $n-2$ times by our linear combination since there are n matrices, and they are hit whenever neither of their 2 blocks are zeroed out. Each $ijji$ component is 'hit' $n-1$ times, for the same reason. In order to make all components share the same coefficient, we subtract each $ijji$ component using a conullity 2 ACT that hits only that component, using the notation from Definition 1.1.2. Now we divide by $n-2$, which is a coefficient shared by every single one of our components, and we have a linear combination which is equal to our original ACT. $\square$

On the next page is an example. We start with a conullity 8 tensor, and as a result we get four conullity 6 ACTs and four conullity 2 ACTs. They have been arranged into pairs in the same way as the equation within the theorem.

$$2R_\psi = R_A + R_B + R_C + R_D - R_\alpha - R_\beta - R_\gamma - R_\delta$$

$$A = \begin{bmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \alpha = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot \end{bmatrix},$$

$$B = \begin{bmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot \end{bmatrix}, \beta = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$C = \begin{bmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot \end{bmatrix}, \gamma = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$D = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -d & \cdot \end{bmatrix}, \delta = \begin{bmatrix} \cdot & a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Note: While this specific proof cannot turn conullity 4 antisymmetrically built ACTs into conullity 2 ACTs of the same build, there is a method by which that can be done.

Theorem 2.2.3. *Antisymmetric Conullity 4 into Conullity 2 Downscaling.* *A conullity 4 antisymmetric ACT can be represented using no more than 10 antisymmetric conullity 2 ACTs.*

Proof. This proof is entirely constructive: I will show that you need only 10 by writing out all 10 of them. Whether this is minimal or not is not known. We start with a rank 4 antisymmetric matrix A:

$$\begin{bmatrix} \cdot & a & \cdot & \cdot \\ -a & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b \\ \cdot & \cdot & -b & \cdot \end{bmatrix}.$$

The ACT R_A has the following components: $R_A(1221) = 3a^2$, $R_A(3443) = 3b^2$, $R_A(1234) = -2ab$, $R_A(1342) = ab$, $R_A(1423) = ab$. The last three of these form a 'Bianchi cluster', where so long as you can force two values, the third one simply falls into place. This is the purpose of the α and β matrices in the expansion below. However, in setting up the cluster, eight $ijki$ components are created. Since our original matrix had zero for all components of the $ijki$ form, we need to reduce all of them to zero. This proof can reduce $ijki$ components to zero at a rate of one ACT per component. That is the reason for the other eight ACTs - two ACTs to handle the Bianchi cluster, and then 8 to get rid of the 8 unwanted byproducts. The $ijji$ components are handled by twisting and careful alignment of coefficients, and do not need their own matrices.

$$\alpha = \begin{bmatrix} . & a & b & . \\ -a & . & . & a \\ -b & . & . & b \\ . & -a & -b & . \end{bmatrix} \qquad \beta = \begin{bmatrix} . & a & . & -b \\ -a & . & b & . \\ . & -a & . & b \\ a & . & -b & . \end{bmatrix}$$

$$\gamma = \begin{bmatrix} . & a & -b & . \\ -a & . & . & . \\ b & . & . & . \\ . & . & . & . \end{bmatrix} \qquad \delta = \begin{bmatrix} . & . & 2b & . \\ . & . & . & . \\ -2b & . & . & b \\ . & . & -b & . \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} . & a & . & . \\ -a & . & 2b & . \\ . & -2b & . & . \\ . & . & . & . \end{bmatrix} \qquad \zeta = \begin{bmatrix} . & a & . & a \\ -a & . & . & . \\ . & . & . & . \\ -a & . & . & . \end{bmatrix}$$

$$\eta = \begin{bmatrix} . & . & . & -2a \\ . & . & . & . \\ . & . & . & b \\ 2a & . & -b & . \end{bmatrix} \qquad \theta = \begin{bmatrix} . & a & . & . \\ -a & . & . & 2a \\ . & . & . & . \\ . & -2a & . & . \end{bmatrix}$$

$$\iota = \begin{bmatrix} . & . & . & . \\ . & . & . & a \\ . & . & . & -b \\ . & -a & b & . \end{bmatrix} \qquad \kappa = \begin{bmatrix} . & . & . & . \\ . & . & -b & . \\ . & b & . & b \\ . & . & -b & . \end{bmatrix}$$

I will now prove that these matrices satisfy the following equality:

$$6R_A = 2R_\alpha + 2R_\beta + 2R_\gamma - R_\delta - R_\epsilon + 2R_\zeta - R_\eta - R_\theta + 2R_\iota + 2R_\kappa.$$

The best way to do this is a sequence of equations for each component. Subscripts will be added to the numbers to indicate which ACT the value originated from. These have no influence on the actual value, and are just for organizational purposes. The coefficients of the values are multiplied by the coefficients applied to the ACT from which the value originated.

$$R_{1221} : 18a_A^2 = 6a_\alpha^2 + 6a_\beta^2 + 6a_\gamma^2 + 6a_\zeta^2 - 3a_\epsilon^2 - 3a_\theta^2$$
$$R_{1331} : 0_A = 6b_\alpha^2 + 6b_\gamma^2 - 12b_\delta^2.$$
$$R_{1441} : 0_A = 6a_\beta^2 + 6a_\zeta^2 - 12a_\eta^2.$$
$$R_{2332} : 0_A = 6b_\beta^2 - 12b_\epsilon^2 + 6b_\kappa^2.$$
$$R_{2442} : 0_A = 6a_\alpha^2 - 12a_\theta^2 + 6a_\iota^2.$$
$$R_{3443} : 18b_A^2 = 6b_\alpha^2 + 6b_\beta^2 - 3b_\delta^2 + 6b_\iota^2 - 3b_\eta^2 + 6b_\kappa^2.$$
$$R_{1234} : -12ab_A = -6ab_\alpha - 6ab_\beta.$$
$$R_{1342} : 6ab_A = 6ab_\alpha.$$
$$R_{1423} : 6ab_A = 6ab_\beta.$$
$$R_{1231} : 0_A = 6ab_\alpha - 6ab_\gamma.$$
$$R_{1241} : 0_A = -6a_\beta^2 + 6a_\zeta^2.$$
$$R_{2142} : 0_A = -6a_\alpha^2 + 6a_\theta^2.$$
$$R_{2132} : 0_A = -6ab_\beta + 6ab_\epsilon.$$
$$R_{3143} : 0_A = -6b_\alpha^2 + 6b_\delta^2.$$
$$R_{3243} : 0_A = -6b_\beta^2 + 6b_\kappa^2.$$
$$R_{4134} : 0_A = -6ab_\beta + 6ab_\eta.$$
$$R_{4234} : 0_A = 6ab_\alpha - 6ab_\iota.$$

Note that there are four components which are not touched by any of the ACTs in the entire equation. This is because the α and β matrices only cover 2 out of the 3 pieces of the Bianchi triple, but the third piece is linearly dependent on the other two so it does not need its own matrix. □

Theorem 2.2.4. ***Clean Antisymmetric Downscaling.*** *If $\psi : V \to V$, $\psi = -\psi^*$, $Rk\psi = 4k > 4$, then it is possible to represent R_ψ using exactly $\binom{2k}{2}$ conullity 4 antisymmetric ACTs, though there is not a generic formula to do so.*

Proof. A JBF-reduced antisymmetric matrix will have $\frac{n}{2}$ $ijji$ components, and when that number is even, it is possible to balance each component without needing conullity 2 ACTs. Each of the Jordan blocks will be present in one fewer matrix than there are $ijji$ components, and each $ijji$ component is tied specifically to one Jordan block. When there is an even number of $ijji$ components, there will be an odd number of matrices. Use twists to control the signs of the matrices so that there is exactly one more positive matrix than there are negative ones. The twists make it so that all the $ijkl$ components are exactly as they should be, and the end result is that the $ijkl$ components are exactly right due to the twists, while the $ijji$ components are also correct thanks to the cancelling-out effect. □

Here is an example. In this case, both C and F are twisted. Since we are starting at rank 8, there will be 2 positive and 1 negative case for each diagonal component, which is most easily arranged by twisting the matrix which contains ac, and then the matrix which contains bd.

$$R_\psi = R_A + R_B - R_C + R_D + R_E - R_F$$

$$A = \begin{bmatrix} . & a & . & . & . & . & . & . \\ -a & . & . & . & . & . & . & . \\ . & . & . & b & . & . & . & . \\ . & . & -b & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}, D = \begin{bmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & b & . & . & . & . \\ . & . & -b & . & . & . & . & . \\ . & . & . & . & . & c & . & . \\ . & . & . & . & -c & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix},$$

$$B = \begin{bmatrix} . & a & . & . & . & . & . & . \\ -a & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & d \\ . & . & . & . & . & . & -d & . \end{bmatrix}, E = \begin{bmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & c & . & . \\ . & . & . & . & -c & . & . & . \\ . & . & . & . & . & . & . & d \\ . & . & . & . & . & . & -d & . \end{bmatrix},$$

$$C = \begin{bmatrix} . & a & . & . & . & . & . & . \\ -a & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & -c & . & . \\ . & . & . & . & c & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}, F = \begin{bmatrix} . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & b & . & . & . & . \\ . & . & -b & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & -d \\ . & . & . & . & . & . & d & . \end{bmatrix}.$$

Theorem 2.2.5. ***Conullity 4 Expansion.*** *If $\psi : V \to V$, $\psi = -\psi^*$, and $Rk\psi = 4$, then there exist four conullity 4 ACTs which sum to R_ψ.*

Proof. The proof of this theorem is constructive. As in the 4:2 downscaling theorem, we start with R_φ equal to

$$\begin{bmatrix} . & a & . & . \\ -a & . & . & . \\ . & . & . & b \\ . & . & -b & . \end{bmatrix}.$$

The four matrices which work for our purposes are:

$$A = \begin{bmatrix} . & 2a & 2a & . \\ -2a & . & . & -b \\ -2a & . & . & 2b \\ . & b & -2b & . \end{bmatrix}, \qquad B = \begin{bmatrix} . & a & a & . \\ -a & . & . & \frac{-b}{2} \\ -a & . & . & b \\ . & \frac{b}{2} & -b & . \end{bmatrix},$$

$$C = \begin{bmatrix} . & a & -a & . \\ -a & . & . & \frac{b}{2} \\ a & . & . & b \\ . & \frac{-b}{2} & -b & . \end{bmatrix}, \qquad D = \begin{bmatrix} . & a & 2a & . \\ -a & . & . & -b \\ -2a & . & . & b \\ . & b & -b & . \end{bmatrix}.$$

The equation they fulfill is as follows:

$$3R_\varphi = R_A - R_B + R_C - R_D.$$

□

Lemma 2.2.6. ***Antisymmetric Upscaling.*** *It is possible to take an antisymmetric ACT of a certain conullity and convert it into a linear combination of 3 ACTs of conullity 2 higher. [1].*

Theorem 2.2.7. ***Antisymmetric Downscaling of Arbitrary Rank.*** *It is always possible to crudely represent an antisymmetric ACT of conullity $2n$ as a linear combination of antisymmetric ACTs of conullity $2k \neq 2$ where $n > k$. This is a crude representation, and the ACTs which are not of the chosen conullity will be of conullity 2. Furthermore, it is always possible to do so using no more than $\binom{n}{n-k} + n$ ACTs in total.*

Proof. This theorem makes two claims. The first claim is that this representation is possible at all. This can be easily shown by repeatedly doing crude downscaling until you are at the desired conullity. The crude downscaling algorithm generates conullity 2 ACTs as its byproduct, which is what the theorem predicted. The second claim is about an upper bound on the number of ACTs required, which is harder to prove. It relies on $ijkl$ component groups. First, find the matrix which represents the antisymmetric ACT and convert it into JBF. Then there will be at most $\binom{n}{2}$ $ijkl$ clusters, along with up to n $ijji$ components. We cover the $ijkl$ clusters in much the same way as the similar proof done on the symmetric case, requiring a total of $\binom{n}{n-k}$ ACTs of conullity 2k. However, we are left with the $ijji$ components, which might be completely in order, but there is no way of telling. As such, we add n ACTs of conullity 2 to set the $ijji$ components into their proper values. ☐

3 Effects on Bounds

3.1 New Results for Previously-Known Values

Corollary 3.1.1. ***A Sharp Value for*** $\mu(3)$***.*** $\mu(3) = 2$.

Proof. We know that for any ACT on a 3-dimensional vector space, that ACT can be represented by two or fewer symmetric ACTs. It was not known whether antisymmetric ACTs would be able to help - if two symmetric ACTs are needed, can you get away with using only one antisymmetric ACT? Since antisymmetric ACTs have only even conullities, and all conullity 2 antisymmetric ACTs are equal to, the answer is no. If you could turn two symmetric ACTs into one antisymmetric ACT (in conullity 2), then that would indicate that the combination of the two symmetric ACTs was itself symmetric, and it did not require two in the first place. ☐

Corollary 3.1.2. ***Chosen-Conullity Spanning Sets.*** *For any $k > 1$, the set of symmetric ACTs of conullity k spans $\mathcal{A}(V)$, and the set of antisymmetric ACTs of conullity $2k$ also spans $\mathcal{A}(V)$.*

Proof. It was shown in Lemma 2.1.1 that any symmetric ACT can be turned into two ACTs of conullity 1 higher. Since we can now turn symmetric ACTs into n ACTs of conullity 1 lower using Theorem 2.1.2, any linear sum of symmetrics can be (inefficiently) rewritten to consist of only matrices of a specific chosen rank. Since the set of all symmetrics is a spanning set ([4]), we have now shown that all symmetrics of a particular conullity are also a spanning set.

Similarly, it was shown in Lemma 2.2.6 that any antisymmetric ACT can be turned into three ACTs of conullity 2 higher. Since we can now downscale antisymmetric ACTs using Theorem 2.2.2, any linear sum of antisymmetrics can be rewritten to consist of only matrices of a specific chosen rank. This requires crudely downscaling to the chosen conullity, and then using Brundan's work to turn the conullity 2 'remainders' into a set of ACTs of the chosen conullity. Since the set of all antisymmetrics is a spanning set ([4]), all antisymmetrics of a particular conullity are also a spanning set. ☐

Theorem 3.1.3. ***The*** μ-η ***Correspondence.***

$$\eta(n) \leq \mu(n)\binom{n}{2}.$$

Proof. If $\mu(n)$ equals some value, it must be the case that it is possible to express any dim n tensor using that number of symmetric and antisymmetric ACTs. Let us assume the worst case for this correspondence, which is that the entire expression is symmetric-only. This is the worst case because if the expression has antisymmetric ACTs, we don't need to convert them into conullity 2 ACTs, which saves us space. The ACTs which make up the combination can be any variety of conullities. Again, we assume the worst case, which is that they all have conullity n. This is the worst case because high-conullity ACTs turn into more conullity 2 ACTs than tensors of low conullity.

Now, we have $\mu(n)$ symmetric ACTs of conullity n. We know that by Theorem 2.1.4, every symmetric ACT R can be expressed as $\binom{k}{2}$ conullity 2 ACTs, where k is the conullity of R. We have $\mu(n) \cdot \binom{n}{2}$ symmetric ACTs of conullity 2. Using Theorem 2.2.1, we can then turn all conullity 2 ACTs into specifically antisymmetric ACTs. Since we always assumed the worst case, we now know that $\eta(n)$ must be less than or equal to the number of ACTs we have, and thus the theorem is proven. □

Note that this theorem does not provide a *better* upper bound for $\eta(n)$. At the moment, $\eta(n)$ is equal to $\frac{n^2(n^2-1)}{12} - \binom{n}{2}$. The correspondence gives us $\frac{n(n+1)}{2}\binom{n}{2} = \frac{n^2(n^2-1)}{4}$, which is slightly more than three times larger. However, there is now a direct link between $\mu(n)$ and $\eta(n)$, which opens the door for several possible future improvements, as described in the Further Questions section.

3.2 Bounds on New Spanning Sets

Due to Corollary 3.1.2, the following concept is well-defined.

Definition 3.2.1. $\mu_{=k}(n)$*. Let $R \in \mathcal{A}(V)$. Then $\mu_{=k}(R)$, based on [7]'s notation of $\mu_k(R)$, is the minimal number of canonical ACTs of conullity k required to represent R as a linear combination. $\mu_{=k}(n)$ is defined as the highest value of $\mu_{=k}(R)$ for all $R \in \mathcal{A}(V)$ where V has dimension n. $\eta_{=k}(n)$ and $\nu_{=k}(n)$ are defined similarly, each with their typical restrictions on what ACTs may be used. Also, $\eta_{=k}(n)$ is only well-defined if k is even.*

Lemma 3.2.2. ***Klinger's Minimal Basis.*** *In [5], it is shown that there exists a basis for dim 4 which guarantees that only fourteen tensor components will be nonzero. The paper itself makes a stronger claim, but we only need this much.*

Theorem 3.2.3. $\eta_{=2}(4) \leq 18$*.*

Proof. This proof uses Lemma 3.2.2's basis. The basis causes six tensor components to be guaranteed to be zero - $R_{1231}, R_{1241}, R_{1341}, R_{2132}, R_{2142}$, and R_{3123}. We perform something similar to Theorem 2.2.3, where we start with the Bianchi components and work from there. Because we need two matrices to address the Bianchi components, eight of the $ijki$-type components will be interfered with, and we can arrange them so that two of our guaranteed-zeroes will be outside of that set. Unlike in Theorem 2.2.3, however, we can't simply balance the $ijji$ components with careful coefficients, and will need to add dedicated ACTs to balance them out. In total, we have six ACTs to balance the $ijji$ components, two to balance the $ijkl$ Bianchi components, and while we would normally have twelve for the $ijki$ components, by arranging them with Klinger's basis, we've guaranteed that two of the $ijki$ components remain equal to zero throughout, and do not need them. As a result, □

4 Further Questions

Conullity 4 antisymmetric-build ACTs are more 'powerful' than other antisymmetrics, because they can invert all of their Bianchi components with a single twist. If it were possible to do the same for antisymmetric-builds of other conullities, then it would be possible to, for example, represent a conullity 12 antisymmetric-build as a sum of conullity 6 ACTs without needing any conullity 2 fragments. Does such a method exist? (If it does, it is almost certain to require complex coefficients.)

How feasible is it to cross-downscale? That is, take an antisymmetric-build and represent it as a sum of symmetrics of conullity one lower, or take a symmetric-build and represent it as a sum of antisymmetrics of conullity one or two lower. Is there a certain form for which the crossover is more efficient?

With downscaling, we not only have the ability to talk about $\nu_{=k}$, but also $\nu_{<k}$. What are the upper bounds of this construction? How does it connect to the antisymmetric downscaling?

The $\mu - \eta$ correspondence introduces a precarious balance that we want to upset. If any of the following five things are shown to be possible, the upper bound for $\eta(n)$ will decrease:

- Turning multiple conullity 2 ACTs into a single antisymmetrically built ACT of conullity 4 or higher. This will improve $\eta(n)$ because the current correspondence generates a massive sea of conullity 2 ACTs, and if they can be compressed, then the value of $\eta(n)$ will drop in accordance with the amount that are compressed away.

- Proving that the upper bound for $\mu(n)$ does not use full conullity in all its matrices. This will improve $\eta(n)$ because if the conullities are not equal to n, we do not need $\binom{n}{2}$ conullity 2 ACTs, but a smaller number. For every ACT that can be shown to be at something other than full rank, we save at least n antisymmetric ACTs from the final total.

- Finding a smaller upper bound for $\nu(n)$. This will improve $\eta(n)$ because they are bound by the correspondence. However much $\nu(n)$ is improved, $\eta(n)$ will improve by the same proportion. However, this will only get us a better value for $\eta(n)$ if there can be a threefold decrease in $\nu(n)$.

- Proving that $\mu(n) < \nu(n)$ for sufficiently high n. If this is shown, then $\eta(n)$ will drop sharply. First, it would mean that there are antisymmetrics in the sum for $\mu(n)$, and we would not have to convert those into conullity 2 ACTs. Second, there would also simply be fewer symmetrics needed to convert into conullity 2 ACTs. Overall, it is hard to estimate how significant this would be, but it would be a massive improvement.

- Finding simultaneous diagonalizability in worst-case $\nu(n)$ linear combinations. This would improve $\eta(n)$ because of the way symmetric downscaling works. When scaling to conullity 2, the only thing that matters is the components present while diagonalized. If you can diagonalize multiple canonical ACTs at the same time in a linear combination, their sum can be downscaled as if it were just a regular symmetric ACT. For example, if five ACTs are simultaneously diagonalizable with each other, it would be possible to remove $\binom{n}{2} \cdot 4$ ACTs from the final total.

5 Acknowledgements and References

I would like to thank Dr. Corey Dunn for his advice and guidance through this project, as well as to thank Dr. Rolland Trapp for our many insightful conversations. I also thank the NSF for the research grant DMS-2050894 and CSU San Bernardino for generously funding this research.

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