# CATEGORIZING EMBEDDED TOTALLY GEODESIC SURFACES WITHIN CHAIN LINKS

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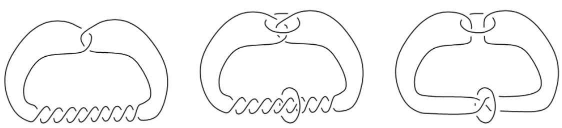
## ABSTRACT

The goal of this research was to completely classify embedded totally geodesic disks with regards to the class of FALs called chain links. We were able to do this by splitting up geodesics disks into those that intersected crossing discs in the circle packing and those that don't. From there, we were able to use the geometry of the circle packings to classify ETGS.

## 1 INTRODUCTION TO FULLY AUGMENTED LINKS

The class studied is that of fully augmented links, or FALs, a class of hyperbolic links. Studied because of their geometric properties , these FALs are obtained by removing all full twist regions and encircling all twist regions with a knot circle, a single unknotted component.



[?]

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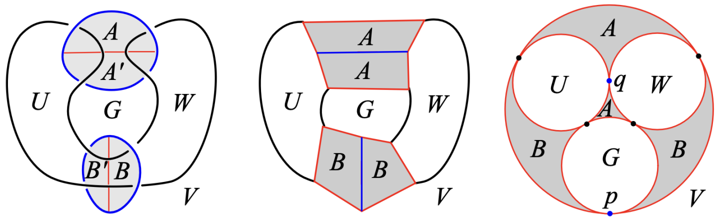
"Each hyperbolic fully augmented link, or FAL, admits a standard cell-decomposition" [?]. Standard cell decomposition are most easily seen when the crossing circles of the FAL are flat. Similar to vertically slicing a piece of pita bread, we slice these crossing circles and flatten them out along the ideal plane. One important thing to note is that because FALs are hyperbolic, they can be decomposed into two identical ideal totally geodesic polyhedra. When we cut along the plane of projection, because these polyhedra are isometric and symmetric about said plane, one only needs to focus on one of those polyhedra (and the top half of those crossing discs).

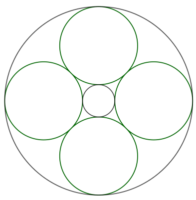



(a) Cell decomposition (b) After slicing 2-cells (c) Circle packing

When we cut along these half crossing discs in a vertical 'pita bread' slice and flatten each part, we will end up with figure 1 a) Then, by shrinking the link components into ideal vertices and transforming all unshaded regions into circles, as seen in c), we will end up with a link specific circle packing!

## 2 GEODESIC DISKS

Geodesic discs are most easily understood by looking through the circle-packing diagram. You can see the example of the circle packing of $P_4$, a length 4 chain link, in Figure 2.



Geodesic discs must be:

1. Made of either circles or straight lines
2. Only goes through points of tangency within the circle packing
3. Either orthogonal everywhere or orthogonal nowhere
4. If orthogonal nowhere, must exit the circle through an adjacent points of tangency or through the 'feet' of the altitude of shaded triangles

To clarify further, we shall say that it goes through an 'inner' point of tangency when it intersects with the 'middle circle'. Similarly, we shall that that it is an 'outer' point of tangency when it intersects with the 'outer circle'.

## 3 AVOIDING CROSSING DISCS

### 3.1 Even Knife Geodesics

The first section we can catalogue are those non-standard geodesic discs that do not intersect with crossing disks. We must explain first, however, the types of punctures that can happen within a geodesic. **Longitudinal** punctures are those made by crossing discs. **Meridional** punctures are those made by the knot components. In other words, we are looking for geodesics that have no longitudinal punctures and only have meridional ones. To save the reader and the researcher much speculation, we can establish an upper limit on the number of meridian punctures a geodesic disk can have.

**Theorem 1.** *A embedded totally geodesic disk can have a maximum of four meridional punctures, two inner and two outer.*

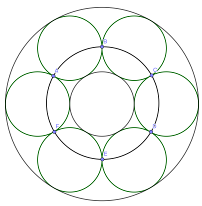
The proof for this theorem is quite simple and can be done a number of ways, however, the 'simplest' lies in basic Euclidean geometry. Recall that you need three points to determine a circle. Thus, if there are 3 (or more) points of tangency with either the outer or the inner ring, two cases happen. Either there does not exist a standard circle that can admit all of the those points or the circle is simply the inner or the outer circle itself.

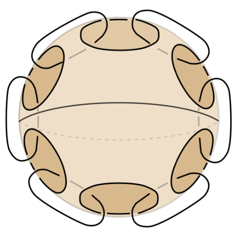
This is useful to know because it cuts down the number of possible geodesic discs to those with less than 4 meridional punctures.

The first type of possible geodesic is called a 'knife geodesic disk'. They are formed using a line and are only found in $P_n$ where $n$, the length of the chain link, is even. We can surmise that these are the only valid geodesic disk that misses crossing discs while including meridional punctures because of the stipulation that the disk must be orthogonal. To expand, because one cannot cross over 'shaded regions' and also that one cannot intersect crossing discs, the only valid 'exit' from either an inner or outer puncture is its opposite along the diameter of the circle. That length can only be crossed orthogonally by a line and thus, it is not necessary to consider the circle case.

### 3.2 Incredibles disk

This is the only valid geodesic disk of its kind, that it is orthogonal to all longitudinal punctures, and can be discovered through some relatively simple geometry. Recall that if circles meet orthogonally, then their tangent lines (which go through the center of the alternate circles) are perpendicular. By using two points that this potential geodesic intersects, both longitudinal, we can find the intersection of those perpendicular lines and can thus identify the center of the potential geodesic disk. This intersection lies squarely at the origin, the only potential center for this geodesic center. Because it is perpendicular to one of those points, seen in the picture below, by origin symmetry, it is perpendicular to all of them. Because there is only one potential center, this is the only geodesic disk that intersects those points of tangency.





## 4 INTERSECTING CROSSING DISCS

The other category of embedded totally geodesic disks within an arbitrary chain link are those that intersect crossing disks. Now, within that category, a disk may either intersect it orthogonally or non-orthogonally.

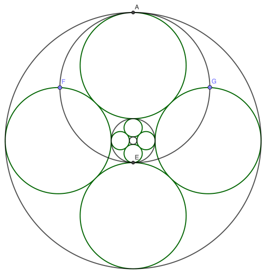
**Theorem 2.** *Let S be a connected, non-standard surface in an FAL complement that intersects at least one crossing disk. Then either*

*i. S meets crossing disks only in separating geodesics and is orthogonal to the standard cell decomposition, or*
*ii. S meets crossing disks only in non-separating geodesics and is nowhere orthogonal to C.*

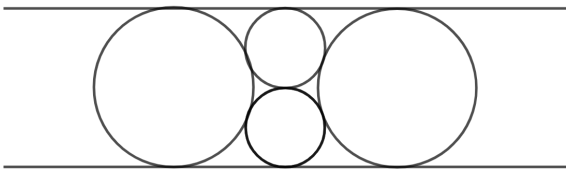
When experimenting with altitudes are geodesic discs, there are only two cases you must consider without loss of generality. Given either an outer or an inner point of tangency, you can either go along the altitude with a straight line or a circle.

### 4.1 Triangular Geodesic Discs

The third case is unique to $P_4$ due to its unusual circle packing. In order to see this, however, one must perform a **möbius transformation** on the circle packing diagram. A möbius transformation is done by taking a specific point of tangency on the circle and sending it to infinity. This means that the circles tangent at that point will become parallel lines through infinity. Another concept we must talk about is the idea of **altitudes**. Intersecting ideal triangles is generally not allowed except when intersecting them through the very middle or the altitude of the triangle. This concept works in tandem with möbius transformations because when one sends a point to infinity, it is much easier to recognize valid altitudes when one end of the line is at infinity. The effect of this is that they must be in the very center of the diagram, also parallel to those circles, now lines.



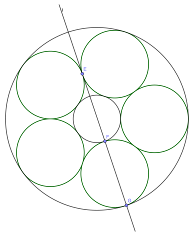

When one takes point P, a representation of $P_4$, and sends that point to infinity, you will get the following circle packing diagram.

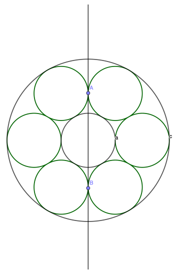


When you see the potential geodesic disk, the only way for it to be valid is if those lines intersecting the ideal triangles run along the altitude. This is much easier to see on the new circle packing diagram. Because the point lies on the point of tangency between two identical circles pressed between two parallel lines, that point of tangency must be the midpoint between those lines and therefore, any parallel line through that point must be the altitude. Therefore, this, called the triangular geodesic disk, is a valid and furthermore, is exclusive to $P_4$. When this möbius transformation is done to a different chain link, say $P_6$, as seen in FIGURE, the circles in the middle will not represent the mid point because they will be different sizes and therefore not equivalent. Therefore, the triangular geodesic disk is exclusive to $P_4$.

### 4.2 Arbitrary Knife Geodesics

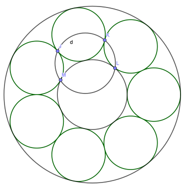
The first case is another variation of our knife geodesics though this one does not depend on your chain's parity as it exists in both odd and even. This knife geodesic intersects two crossing discs and runs along 4 altitudes given any chain link. A picture of this, in both cases, can be seen below.

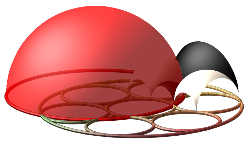




### 4.3 Butterfly and Scoop Geodesic Disks

The second type works for arbitrary length of the chain link and is identical regardless of the parity of the length. It is defined by having two longitudinal punctures and two meridional punctures, both on the inner points of tangency. When this geodesic disk is looked at in hyperbolic space, it can be seen that it rides the seam of two different spheres, therefore splitting them apart.





Also working for arbitrary length, this geodesic disk is defined by also having two longitudinal punctures and two meridional punctures, however, these are now on the outer points of tangency.