Structure Groups of Algebraic Curvature Tensors Containing Rotation Actions

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Abstract

In this paper we examine Algebraic Curvature Tensors and Model Spaces over dimension 3 whose structure groups contain rotation actions. In particular, the rotation actions we analyze are the circular action $\mathbb{S}^1$ and the hyperbolic boost transformation. We also examine structure groups that contain elements of finite order $k \neq 2$. Since every structure group is a Lie group, we use the structure of its Lie algebra to obtain information about the structure group.

1 Introduction

We first start by providing some definitions. Let V be a real finite dimensional vector space.

Definition 1.1. Let $x, y, z, w \in V$ and let $R \in \otimes^4 V^*$. We say R is an Algebraic Curvature Tensor (or ACT) if it has the following properties:

$$R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z) = R(z,w,x,y), \quad \text{and}$$
$$R(x,y,z,w) + R(x,w,y,z) + R(x,z,w,y) = 0.$$

We let $\mathcal{A}(V)$ denote the set of all ACTs over V where $\dim V = n$.

Next, let $A \in \mathrm{GL}(n,\mathbb{R})$. We define the *precomposition* of A with R, denoted A^*R, as

$$A^*R(x,y,z,w) = R(Ax,Ay,Az,Aw).$$

In general, it is not the case that $A^*R(x,y,z,w) = R(x,y,z,w)$. Although when equality does hold, the set of such $A \in \mathrm{GL}(n,\mathbb{R})$ form a group under composition.

Definition 1.2. We define the structure group of R, denoted G_R, as the group

$$G_R = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^*R(x,y,z,w) = R(x,y,z,w) \text{ for all } x,y,z,w \in V\}.$$

Structure groups are important since given $R \in \mathcal{A}(V)$, its structure group G_R is precisely the group of transformations of V that fix R. Previous work on structure groups include Obeidin's work in the case that $\dim V = 3$, in which he classifies all possible G_R for $R \in \mathcal{A}(V)$ [Obe12].

Now suppose V is equiped with an inner product $\langle \cdot, \cdot \rangle$. When V has an inner product, we say the triple $\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, R)$ is a *model space*. For $A \in \mathrm{GL}(n,\mathbb{R})$, we can also precompose A with the inner product, defined as

$$A^*\langle x, y \rangle = \langle Ax, Ay \rangle.$$

As before, it is not always the case that $A^*\langle x, y \rangle = \langle x, y \rangle$, but when such equality holds in addition to $A^*R = R$, the set of such matrices form a group under composition.

Definition 1.3. Let $\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. We define the structure group of $\mathfrak{M}$, denoted $G_{\mathfrak{M}}$, as the group

$$G_{\mathfrak{M}} = \{A \in \mathrm{GL}(n, \mathbb{R}) \ | \ A^*R = R \text{ and } A^*\langle x, y \rangle = \langle x, y \rangle \text{ for all } x, y \in V\}.$$

In both cases of G_R and $G_{\mathfrak{M}}$, the structure group is actually a *Lie Group*.

Definition 1.4. A Lie Group G is a group that is also a real smooth manifold such that the two maps $G \times G \to G$ given by

$$g \cdot h \mapsto gh \text{ and } g \mapsto g^{-1}$$

are smooth. [Sep06]

Given a structure group, it is helpful to uncover some information about the structure group. Since every structure group is a Lie group, which is also a manifold, we can calculate the dimension of the Lie group. In order to do so, we can calulate the dimension of the *Lie Algebra* associated to the structure group.

Definition 1.5. For a Lie group G, its associated Lie Algebra $\mathfrak{g}$ is the tangent space of G at the identity of G.

One method to find the dimension of a Lie group G is to consider some path $g(t)$ through the identity $I \in G$ such that $g(t) \in G$ for all t in some interval J and $g(p) = I$ for some $p \in J$. Without loss of generality, suppose $g(0) = I$. Then one can evaluate $g'(p)$ for some $p \in J$, which is the tangent space of G at point $g(p) \in G$. So for some path $g(t)$ such that $g(0) = I$, the tangent space at the identity is the space given by $g'(0)$, or in other words, the Lie algebra $\mathfrak{g}$ of G. We are then able to find basis vectors for $\mathfrak{g}$, in which case the number of basis vectors is also the dimension of $\mathfrak{g}$ and hence the dimension of the group G. For notational purposes, we will let $\mathfrak{g}_R$ and $\mathfrak{g}_{\mathfrak{M}}$ denote the Lie algebra of G_R and $G_{\mathfrak{M}}$, respectively.

We now give an outline of this paper. In Section 2, we work over V of dimension 3 and find all $R \in \mathcal{A}(V)$ that are preserved by the rotation action $\mathbb{S}^1$ by using the method above. As a result of this, we show that for any G_R containing an element of finite order $k > 2$, R is preserved by $\mathbb{S}^1$ as well. In Section 3, we work over V of dimension 3 again and find all $R \in \mathcal{A}(V)$ that are preserved by a hyperbolic boost using the method above as well. In Section 4, we consider model spaces $\mathfrak{M}$ over dimension 3 that are preserved by $\mathbb{S}^1$ and a hyperbolic boost. At the end of each section, we provide a summary of the results found in that section. We then conclude with open questions that can lead to future work. Lastly, we state acknowledgements and refrences cited in this paper.

2 Structure groups containing an $\mathbb{S}^1$ action

We first describe all ACTs over dimension 3. Let V be a 3-dimensional vector space with basis $\beta_V = \{e_1, e_2, e_3\}$ and let $\mathcal{A}(V)$ be as in Definition 1.1. It is known that $\mathcal{A}(V)$ is a 6-dimensional vector space where each component is determined by the output of each possible combination of inputs of basis vectors up to the symmytries in Definition 1.1. For notation, we let

$$R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell)$$

for $1 \le i, j, k, \ell \le 3$. For $R \in \mathcal{A}(V)$ with $\dim V = 3$, the six components of R are

$$R_{1221}, R_{1331}, R_{2332}, R_{1231}, R_{2132}, R_{3123}.$$

For convinence, we define each of the six curvature entries as follows:

$$\begin{aligned} R_1 &= R_{1221}, \\ R_2 &= R_{1331}, \\ R_3 &= R_{2332}, \\ R_4 &= R_{1231}, \\ R_5 &= R_{2132}, \\ R_6 &= R_{3123}. \end{aligned}$$

We will refer to these entries as above throughout the rest of this paper. We also provide one more definition before our analysis.

Definition 2.1. Let $S^2(V^*)$ denote the set of all bilinear forms on V. Given $\phi \in S^2(V^*)$, we define

$$R_\phi(x,y,z,w) = \phi(x,w)\phi(y,z) - \phi(x,z)\phi(y,w).$$

Since ϕ is bilinear, ϕ is determined entirely by the entries of the basis vectors for V into ϕ. If $\beta_V = \{e_1, \dots, e_n\}$ is a basis for V, we let $\phi_{ij} = \phi(e_i, e_j)$ and let

$$[\phi] = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{1n} & \cdots & \phi_{nn} \end{bmatrix}$$

denote the array of entries of ϕ for all possible combinations of basis vectors for V, up to symmetry of ϕ.

2.1 A classification of all ACTs in dimension 3 preserved by $\mathbb{S}^1$

We will derive all possible ACTs over dimension 3 whose structure groups contain an $\mathbb{S}^1$ action. Let $R \in \mathcal{A}(V)$ be an ACT over V of dimension 3 with basis $\beta_V = \{e_1, e_2, e_3\}$ and suppose the structure group of R contains $\mathbb{S}^1$. Without loss of generality, suppose this $\mathbb{S}^1$ action acts on the plane spanned by e_2 and e_3. Such an action can be represented by the following matrix

$$A_{\theta,\lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $\lambda \in \mathbb{R} - \{0\}$ is fixed and $\theta \in [0, 2\pi)$. By definition, we must have $A^*_{\theta,\lambda} R_i = R_i$ for all $i = 1, \dots, 6$. Also, since we are assuming the structure group contains $\mathbb{S}^1$, we must have $A^*_{\theta,\lambda} R_i = R_i$ for all $\theta \in [0, 2\pi)$. By the calulations of Beneish [Ben13], we arrive at the following equalities for $A^*_{\theta,\lambda} R_i$:

$$R_1 = A^*_{\theta,\lambda} R_1 = \lambda^2 \cos^2\theta R_1 + \lambda^2 \sin^2\theta R_2 + 2\lambda^2 \cos\theta \sin\theta R_4 \tag{1}$$
$$R_2 = A^*_{\theta,\lambda} R_2 = \lambda^2 \sin^2\theta R_1 + \lambda^2 \cos^2\theta R_2 - 2\lambda^2 \cos\theta \sin\theta R_4 \tag{2}$$
$$R_3 = A^*_{\theta,\lambda} R_3 = (\sin^2\theta + \cos^2\theta)^2 R_3 \tag{3}$$
$$R_4 = A^*_{\theta,\lambda} R_4 = -\lambda^2 \cos\theta \sin\theta R_1 + \lambda^2 \cos\theta \sin\theta R_2 + \lambda^2(\cos^2\theta - \sin^2\theta) R_4 \tag{4}$$
$$R_5 = A^*_{\theta,\lambda} R_5 = \lambda \cos\theta R_5 - \lambda \sin\theta R_6 \tag{5}$$
$$R_6 = A^*_{\theta,\lambda} R_6 = \lambda \sin\theta R_5 + \lambda \cos\theta R_6. \tag{6}$$

Claim 2.2. *We claim $R_1 = R_2$ is free, R_3 is free, and $R_4 = R_5 = R_6 = 0$.*

Proof. Since $(\sin^2\theta + \cos^2\theta)^2 = 1^2 = 1$, we have that the (3) holds for all θ. There are two possible cases, depending on if $\lambda \in \mathbb{R} - \{-1, 0, 1\}$ or if $\lambda = \pm 1$. Note that $\lambda \neq 0$ since if $\lambda = 0$, $A_{\theta,\lambda}$ would be degenerate.

1. Case $\lambda \in \mathbb{R} - \{-1, 0, 1\}$: Suppose $\lambda \in \mathbb{R} - \{-1, 0, 1\}$. Since Equations (1)$-$(6) must hold for all $\theta \in [0, 2\pi)$, they all must hold for $\theta = 0$. If $\theta = 0$, we get the equations

$$\begin{aligned} R_1 &= \lambda^2 R_1, \\ R_2 &= \lambda^2 R_2, \\ R_3 &= R_3, \\ R_4 &= \lambda^2 R_4, \\ R_5 &= \lambda R_5, \\ R_6 &= \lambda R_6. \end{aligned}$$

Since $\lambda \neq \pm 1$, we must have that $R_1 = R_2 = R_4 = R_5 = R_6 = 0$.

2. Case $\lambda = \pm 1$: Next, we look at if $\lambda = \pm 1$. For (5) and (6), if $\lambda = -1$ and $\theta = 0$, we get

$$R_5 = -R_5$$
$$R_6 = -R_6,$$

so it must be the case that $R_5 = R_6 = 0$. Next, suppose $\lambda = 1$, We can multiply (5) and (6) by $\cos\theta$ and $\sin\theta$ respectively. We then get

$$\cos\theta R_5 = \cos^2\theta R_5 - \cos\theta\sin\theta R_6$$
$$\sin\theta R_6 = \sin^2\theta R_5 + \cos\theta\sin\theta R_6.$$

If we add these two equations, we get $\cos\theta R_5 + \sin\theta R_6 = R_5$. So it must be the case that

$$\cos\theta R_5 + \sin\theta R_6 = \cos\theta R_5 - \sin\theta R_6,$$

or equivalently,

$$2\sin\theta R_6 = 0.$$

Since this equality must hold for all $\theta \in [0, 2\pi)$, it must hold when $\sin\theta \neq 0$ and hence $R_6 = 0$. Substituting this into (5), we see that

$$R_5 = \cos\theta R_5,$$

which must also hold for all θ and hence when $\cos\theta \neq 1$. So, we must have that $R_5 = 0$ as well. In either case of $\lambda = \pm 1$, we get $R_5 = R_6 = 0$.

Next, for (1),(2) and (4), in either case of $\lambda = 1$ or $\lambda = -1$, the equations end up being equivalent. Also, these equations must hold for all $\theta \in [0, 2\pi)$. In particular, they must hold for all $\theta \in [0, 2\pi) - \{0, \pi\}$. For the sake of our calculations, suppose $\theta \notin \{0, \pi\}$. If we subtract (2) from (1), we get

$$R_1 - R_2 = (\cos^2\theta - \sin^2\theta)R_1 - (\cos^2\theta - \sin^2\theta)R_2 + 4\cos\theta\sin\theta R_4 = (\cos 2\theta)(R_1 - R_2) + 2\sin 2\theta R_4.$$

If we solve for $R_1 - R_2$, we get

$$(1 - \cos 2\theta)(R_1 - R_2) = 2\sin 2\theta R_4,$$

so

$$R_1 - R_2 = \frac{2\sin 2\theta}{(1 - \cos 2\theta)} R_4.$$

Since $\theta \notin \{0, \pi\}$, $\cos 2\theta \neq 1$ and hence $(1 - \cos 2\theta) \neq 0$. If we simplify (4), we see

$$R_4 = \left(-\frac{1}{2}\sin 2\theta\right)(R_1 - R_2) + \cos 2\theta R_4.$$

Then by plugging in $R_1 - R_2 = \frac{2\sin 2\theta}{(1-\cos 2\theta)} R_4$, we get

$$R_4 = \left(-\frac{1}{2}\sin 2\theta\right)\left(\frac{2\sin 2\theta}{(1 - \cos 2\theta)}\right) R_4 + \cos 2\theta R_4.$$

This simplifies to

$$R_4 = \left(-\frac{\sin^2\theta}{1 - \cos 2\theta}\right) R_4 + \cos 2\theta R_4,$$

or equivalently,

$$R_4 = -\frac{1 - \cos^2 2\theta}{1 - \cos 2\theta} R_4 + \cos 2\theta R_4 = -\frac{(1 - \cos 2\theta)(1 + \cos 2\theta)}{1 - \cos 2\theta} R_4 + \cos 2\theta R_4.$$

Then simplifying more gives us

$$R_4 = -(1 + \cos 2\theta)R_4 + \cos 2\theta R_4 = -R_4 - \cos 2\theta R_4 + \cos 2\theta R_4 = -R_4.$$

Since $R_4 = -R_4$, we must have $R_4 = 0$. Now, (1) and (2) become

$$R_1 = \cos^2\theta R_1 + \sin^2\theta R_2$$

and

$$R_2 = \sin^2\theta R_1 + \cos^2\theta R_2,$$

respectively. Then

$$R_1 - R_2 = (\cos^2\theta - \sin^2\theta)(R_1 - R_2) = (\cos 2\theta)(R_1 - R_2).$$

Again since $\cos 2\theta \neq 1$ for $\theta \notin \{0, \pi\}$, we get that $R_1 - R_2 = 0$ and hence $R_1 = R_2$.

In all cases of $\lambda \in \mathbb{R} - \{0\}$, we have $R_1 = R_2$, R_3 is free, and $R_4 = R_5 = R_6 = 0$, so we can label each curvature component at $R_1 = R_2 = \alpha$, $R_3 = \beta$, and $R_4 = R_5 = R_6 = 0$ for some $\alpha, \beta \in \mathbb{R}$. Thus, the subspace of $\mathcal{A}(V)$ whose elements are preserved by $\mathbb{S}^1$ is the set of curvature tensors whose components are of the following form:

$$R_1 = R_2 = \alpha$$
$$R_3 = \beta$$
$$R_4 = R_5 = R_6 = 0$$

for some $\alpha, \beta \in \mathbb{R}$. □

Throughout the rest of the paper, let $S \subseteq \mathcal{A}(V)$ be the subspace spanned by ACTs of the form above. For some ACT $R \in S$, we have just shown that the structure group G_R of R contains $\mathbb{S}^1$ as at least a subgroup of G_R, but G_R can contain more elements than just those of $\mathbb{S}^1$. We now examine the different possible cases for R and G_R, depending on all possible choices of $\alpha, \beta \in \mathbb{R}$.

Calculation 2.3. For some $R \in \mathcal{A}(V)$ such that $R_4 = R_5 = R_6 = 0$, we will calculate $A^*R = R$ for some arbitrary $A \in G_R$ below.

Firstly, let

$$A = \begin{pmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{pmatrix} \in G_R.$$

Then, we must have $A^*R_i = R_i$ for $i = 1, \ldots, 6$. This gives us the equations

$$R_1 = A^*R_1 = (x_1x_5 - x_2x_4)^2R_1 + (x_1x_6 - x_3x_4)^2R_2 + (x_2x_6 - x_3x_5)^2R_3$$
$$R_2 = A^*R_2 = (x_1x_8 - x_2x_7)^2R_1 + (x_1x_9 - x_3x_7)^2R_2 + (x_2x_9 - x_3x_8)^2R_3$$
$$R_3 = A^*R_3 = (x_4x_8 - x_5x_7)^2R_1 + (x_4x_9 - x_6x_7)^2R_2 + (x_5x_9 - x_6x_8)^2R_3$$
$$R_4 = A^*R_4 = (x_1x_5 - x_2x_4)(x_1x_8 - x_2x_7)R_1 + (x_1x_6 - x_3x_4)(x_1x_9 - x_3x_7)R_2 + (x_2x_6 - x_3x_5)(x_2x_9 - x_3x_8)R_3$$
$$R_5 = A^*R_5 = -(x_1x_5 - x_2x_4)(x_4x_8 - x_5x_7)R_1 - (x_1x_6 - x_3x_4)(x_4x_9 - x_6x_7)R_2 - (x_2x_6 - x_3x_5)(x_5x_9 - x_6x_8)R_3$$
$$R_6 = A^*R_6 = (x_1x_8 - x_2x_7)(x_4x_8 - x_5x_7)R_1 + (x_1x_9 - x_3x_7)(x_4x_9 - x_6x_7)R_2 + (x_2x_9 - x_3x_8)(x_5x_9 - x_6x_8)R_3.$$

Since we have a multilinear system of equations of R_i in terms of variables x_j, we can take the partial derivative of each x_j to obtain the Jacobian matrix $\left(\frac{\partial R_i}{\partial x_j}\right)$. We provide one example of this by differentiating R_1 with respect to x_1 implicitly. Note that since each $R_i \in \mathbb{R}$ is a constant, the derivative of every R_i with respect to any variable is 0. To avoid confusion, the R_1 in $\frac{\partial R_1}{\partial x_1}$ on the left hand side is a function of multiple x_j, while the R_1 multiplied by many x_j on the right hand side is the constant R_1. Observe that

$$0 = \frac{\partial R_1}{\partial x_1} = 2x_5(x_1x_5 - x_2x_4)R_1.$$

We repeat this process by taking the partial derivative of every R_i with respect to each x_j implicitly, giving us $\left(\frac{\partial R_i}{\partial x_j}\right)$. Next, we can use this Jacobian matrix to find the dimension of G_R. Since G_R is a Lie group, we can find its dimension by finding the dimension of the Lie algebra $\mathfrak{g}_R$ of G_R. So, let $t \in J$ be an interval containing 0 and let

$$A(t) = \begin{pmatrix} x_1(t) & x_4(t) & x_7(t) \\ x_2(t) & x_5(t) & x_8(t) \\ x_3(t) & x_6(t) & x_9(t) \end{pmatrix}$$

such that $A(0) = I_3$, the identity 3×3 matrix. In other words, we now have a continuous path $A(t)$ in G_R such that $A(t) \in G_R$ for all $t \in J$ and $A(0) = I_3$. We can modify $\left(\frac{\partial R_i}{\partial x_j}\right)$ by substituting every x_j coefficient in $\left(\frac{\partial R_i}{\partial x_j}\right)$ with $x_j(t)$ as a function of t. If we use the fact that $R_4 = R_5 = R_6 = 0$ and evaluate each $x_j(t)$ at $t = 0$, we get following modified Jacobian matrix $\left(\frac{\partial R_i}{\partial x_j}\right)_{|t=0}$:

$$\left(\frac{\partial R_i}{\partial x_j}\right)_{|t=0} = \begin{pmatrix} 2R_1 & 0 & 0 & 0 & 2R_1 & 0 & 0 & 0 & 0 \\ 2R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2R_2 \\ 0 & 0 & 0 & 0 & 2R_3 & 0 & 0 & 0 & 2R_3 \\ 0 & 0 & 0 & 0 & 0 & R_2 & 0 & R_1 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & R_1 & 0 & 0 \\ 0 & R_3 & 0 & R_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we can use the chain rule by multiplying the modified Jacobian matrix $\left(\frac{\partial R_i}{\partial x_j}\right)_{|t=0}$ by $\left(\frac{\partial x_j}{\partial t}\right)_{|t=0}$ to take the derivative of each R_i with respect to t and evaluating at $t = 0$. For notational purposes, let $x_j' = x_j'(0)$. Then, we see

$$\left(\frac{\partial R_i}{\partial t}\right)_{|t=0} = \begin{pmatrix} 2R_1 & 0 & 0 & 0 & 2R_1 & 0 & 0 & 0 & 0 \\ 2R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2R_2 \\ 0 & 0 & 0 & 0 & 2R_3 & 0 & 0 & 0 & 2R_3 \\ 0 & 0 & 0 & 0 & 0 & R_2 & 0 & R_1 & 0 \\ 0 & 0 & R_3 & 0 & 0 & 0 & R_1 & 0 & 0 \\ 0 & R_3 & 0 & R_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_9' \end{pmatrix} = \begin{pmatrix} 2R_1(x_1' + x_5') \\ 2R_2(x_1' + x_9') \\ 2R_3(x_5' + x_9') \\ R_2 x_6' + R_1 x_8' \\ R_3 x_3' + R_1 x_7' \\ R_3 x_2' + R_2 x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad (\clubsuit)$$

This concludes Calculation 2.3. □ If we substitute $R_1 = R_2 = \alpha$ and $R_3 = \beta$ into the equations above, we get the following equations:

$$\begin{pmatrix} 2\alpha(x_1' + x_5') \\ 2\alpha(x_1' + x_9') \\ 2\beta(x_5' + x_9') \\ \alpha x_6' + \alpha x_8' \\ \beta x_3' + \alpha x_7' \\ \beta x_2' + \alpha x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(a)}$$

We will now look at the four possible cases: $\alpha = 0$ and $\beta = 0$, $\alpha = 0$ and $\beta \neq 0$, $\alpha \neq 0$ and $\beta = 0$, $\alpha \neq 0$ and $\beta \neq 0$.

1. Case $\alpha = \beta = 0$: Let $R \in S$ as above be such that $\alpha = \beta = 0$. Then all of the equations in (a) hold for all choices of x_j and hence each x_j is free for all $j = 1, \ldots, 9$. Hence, $\dim G_R = 9$ and so $G_R = \mathrm{GL}(3, \mathbb{R})$. In this case, $R = 0$, the zero curvature tensor.

2. Case $\alpha = 0$ and $\beta \neq 0$: Let $R \in S$ such that $\alpha = 0$ and $\beta \neq 0$. Then the equations in (a) become

$$\begin{pmatrix} 0 \\ 0 \\ 2\beta(x_5' + x_9') \\ 0 \\ \beta x_3' \\ \beta x_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we see that $x_5 = -x_9, x_2 = x_3 = 0$ and $x_1, x_4, x_6, x_7, x_8, x_9$ are free. Hence, $\dim G_R = 6$ and a basis $\beta_{\mathfrak{g}_R}$ for $\mathfrak{g}_R$ is

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Observe that $R_{1221} = R_{1331} = 0$, so $e_1 \in \ker R$. Since R has non-trivial kernel, we can decompose R as $R = R_1 \oplus R_2$ and V as $V = V_1 \oplus V_2$, where $V_1 = \operatorname{span}\{e_1\}$ and $V_2 = \operatorname{span}\{e_2, e_3\}$ and each $R_i \in \mathcal{A}(V_i)$. [DFP11]

3. Case $\alpha \neq 0$ and $\beta = 0$: Let $R \in S$ such that $\alpha \neq 0$ and $\beta = 0$. Then the equations in (a) become

$$\begin{pmatrix} 2\alpha(x_1' + x_5') \\ 2\alpha(x_1' + x_9') \\ 0 \\ \alpha x_6' + \alpha x_8' \\ \alpha x_7' \\ \alpha x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we can see that $x_5 = x_9 = -x_1$, $x_6 = -x_8$, $x_4 = x_7 = 0$ and x_2, x_3 are free. So we have 4 free variables x_1, x_2, x_3, x_8 which give rise to a basis $\beta_{\mathfrak{g}_R}$ of $\mathfrak{g}_R$ given by:

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

In this case, we have that $R = R_{\phi_1} + R_{\phi_2}$ where $\phi_1, \phi_2 \in S^2(V^*)$ and

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

4. Case $\alpha \neq 0$ and $\beta \neq 0$: Let $R \in S$ such that $\alpha \neq 0$ and $\beta \neq 0$. Then our equations in (a) become

$$\begin{pmatrix} 2\alpha(x_1' + x_5') \\ 2\alpha(x_1' + x_9') \\ 2\beta(x_5' + x_9') \\ \alpha x_6' + \alpha x_8' \\ \beta x_3' + \alpha x_7' \\ \beta x_2' + \alpha x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we solve for each variable, we see that $x_1 = x_5 = x_9 = 0$, $x_6 = -x_8$, $x_2 = -\frac{\alpha}{\beta} x_4$, and $x_3 = -\frac{\alpha}{\beta} x_7$. Thus we have three free variables, given by x_4, x_7, x_8. These then give us three basis vectors for basis $\beta_{\mathfrak{g}_R}$ of $\mathfrak{g}_R$, given by:

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -\frac{\alpha}{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{\alpha}{\beta} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

In this case, we have that $R = \operatorname{sgn}(\beta) R_\phi$ for some $\phi \in S^2(V^*)$ given by

$$\phi = \begin{pmatrix} \operatorname{sgn}(\beta)\alpha/\sqrt{|\beta|} & 0 & 0 \\ 0 & \sqrt{|\beta|} & 0 \\ 0 & 0 & \sqrt{|\beta|} \end{pmatrix},$$

where $\operatorname{sgn}(\beta)$ denotes the sign function of β. It is known that $G_{R_\phi} = G_\phi$, and in particular, $G_\phi = O(p, q)$ where (p, q) is the signature of ϕ [DFP11]. We let p represent the number of negative entries in ϕ and let q represent the number of positive entries of ϕ. So, since $\phi_{22}, \phi_{33} > 0$, we have $G_R = O(1, 2)$ or $G_R = O(3)$ depending on the signature of ϕ.

This is now a complete classification of all $R \in \mathcal{A}(V)$ over $\dim V = 3$ that are preserved by $\mathbb{S}^1$. To summarize, the possible dimensions of G_R are 9, 6, 4, and 3. In the case that $\dim G_R = 9$, we have that $G_R = \mathrm{GL}(3, \mathbb{R})$. In the case that $\dim G_R = 3$, we have that $G_R = O(3)$ or $G_R = (1, 2)$.

2.2 Elements of G_R of finite order in dimension 3

Let $R \in \mathcal{A}(V)$ where $\dim V = 3$ and suppose $A \in G_R$ has finite order $k > 2$. We focus on elements of order $k > 2$ since elements of $\mathrm{GL}(n, \mathbb{R})$ of order 2 arise from reflections or products of reflections [Koo03]. In the case that $\dim V = 3$, Williams has already detailed some structure groups whose elements have order 2 [Wil19]. In Koo's paper, Koo describes all elements of $\mathrm{GL}(n, \mathbb{R})$ of finite order. According to Koo, if A has finite order $k > 2$, then it must be of the form

$$A = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $\varepsilon = \pm 1$, $\theta = \frac{2a\pi}{b}$ for some $a, b \in \mathbb{Z}$ with $b > 2$, and $k = b$ or $k = \mathrm{lcm}\{2, b\}$, depending on if $\varepsilon = 1$ or $\varepsilon = -1$, respectively. We suppose $b > 2$, so we have that $\theta \neq \pi$. Since A is of the same form as $A_{\theta, \pm 1}$ from Section 2.1, one might realize there are similarities between elements of finite order $k > 2$ and elements of $\mathbb{S}^1$. This leads us to the following theorem.

THEOREM 2.4. *Let R be an ACT over dimension 3 such that there exists $A \in G_R$ of order $k > 2$. Then $\mathbb{S}^1$ is a subgroup of G_R.*

Proof. Let $R \in \mathcal{A}(V)$ such that $\dim V = 3$ and suppose G_R contains an element of finite order $k > 2$. Let $A \in G_R$ be such an elements. Then by Koo, A is of the form

$$A = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

where $\theta = 2\pi\frac{a}{b}$ for some $a, b \in \mathbb{Z}$, $b > 2$, and $\varepsilon = \pm 1$ [Koo03]. Also, the order k of A is $k = b$ or $k = \mathrm{lcm}\{2, b\}$, depending on if $\varepsilon = 1$ or $\varepsilon = -1$, respectively. Since $A \in G_R$, we must have $A^* R_i = R_i$ for all i. If we expand $A^* R_i = R_i$, we get the following system of equations:

$$R_1 = A^* R_1 = \cos^2\theta R_1 + \sin^2\theta R_2 + 2\cos\theta\sin\theta R_4 \tag{7}$$
$$R_2 = A^* R_2 = \sin^2\theta R_1 + \cos^2\theta R_2 - 2\cos\theta\sin\theta R_4 \tag{8}$$
$$R_3 = A^* R_3 = (\sin^2\theta + \cos^2\theta)^2 R_3 \tag{9}$$
$$R_4 = A^* R_4 = -\cos\theta\sin\theta R_1 + \cos\theta\sin\theta R_2 + (\cos^2\theta - \sin^2\theta) R_4 \tag{10}$$
$$R_5 = A^* R_5 = \varepsilon\cos\theta R_5 - \varepsilon\sin\theta R_6 \tag{11}$$
$$R_6 = A^* R_6 = \varepsilon\sin\theta R_5 + \varepsilon\cos\theta R_6. \tag{12}$$

We can see that Equation (9) always holds. We can follow the calculations in Case 2 of Claim 2.2 identically since $\cos 2\theta \neq 1$ because $\theta \notin \{0, \pi\}$. Thus, we conclude that $R_1 = R_2$, R_3 is free, and $R_4 = R_5 = R_6 = 0$. Thus, $R \in S$ and hence $\mathbb{S}^1$ must preserve R as well.

□

Theorem 2.4 tells us that in the case $\dim V = 3$, if there exists $A \in G_R$ of finite order $k > 2$ that preserves R, then $\mathbb{S}^1$ must also preserve R.

3 Hyperbolic Boost in dimension 3

We now focus our attention to the *Hyperbolic Boost* action in dimension 3. This action can be described by the matrix

$$B_{\theta,\lambda} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \cosh\theta & -\sinh\theta \\ 0 & -\sinh\theta & \cosh\theta \end{pmatrix},$$

where $\lambda \in \mathbb{R} - \{0\}$ is fixed and $\theta \in \mathbb{R}$. Suppose there exists $R \in \mathcal{A}(V)$ over $\dim V = 3$ such that $B_{\theta,\lambda} \in G_R$ for all $\theta \in \mathbb{R}$. Then we must have $B_{\theta,\lambda}^* R_i = R_i$ for all i, which gives us the following equations:

$$R_1 = B_{\theta,\lambda}^* R_1 = \lambda^2 \cosh^2\theta R_1 + \lambda^2 \sinh^2\theta R_2 - 2\lambda^2 \cosh\theta \sinh\theta R_4 \tag{13}$$
$$R_2 = B_{\theta,\lambda}^* R_2 = \lambda^2 \sinh^2\theta R_1 + \lambda^2 \cosh^2\theta R_2 - 2\lambda^2 \cosh\theta \sinh\theta R_4 \tag{14}$$
$$R_3 = B_{\theta,\lambda}^* R_3 = (\sinh^2\theta - \cosh^2\theta)^2 R_3 \tag{15}$$
$$R_4 = B_{\theta,\lambda}^* R_4 = -\lambda^2 \cosh\theta \sinh\theta R_1 - \lambda^2 \cosh\theta \sinh\theta R_2 + \lambda^2(\cosh^2\theta + \sinh^2\theta) R_4 \tag{16}$$
$$R_5 = B_{\theta,\lambda}^* R_5 = \lambda \cosh\theta R_5 + \lambda \sinh\theta R_6 \tag{17}$$
$$R_6 = B_{\theta,\lambda}^* R_6 = \lambda \sinh\theta R_5 + \lambda \cosh\theta R_6. \tag{18}$$

Claim 3.1. *We claim that $R_1 = -R_2$ is free, R_3 is free, and $R_4 = R_5 = R_6 = 0$.*

Proof. As in Claim 2.2, there are two cases depending on if $\lambda \in \mathbb{R} - \{-1, 0, 1\}$ and if $\lambda = \pm 1$. Note that these Equations (13)$-$(18) must hold for all $\theta \in \mathbb{R}$. Observe that equation (15) holds for all $\theta \in \mathbb{R}$ since $\cosh^2\theta - \sinh^2\theta = 1$ for all $\theta \in \mathbb{R}$.

1. Case $\lambda \in \mathbb{R} - \{-1, 0, 1\}$: Suppose $\lambda \in \mathbb{R} - \{-1, 0, 1\}$. Then Equations (13)$-$(18) must hold for all $\theta \in \mathbb{R}$. If $\theta = 0$, we have

$$\begin{aligned} R_1 &= \lambda^2 R_1 \\ R_2 &= \lambda^2 R_2 \\ R_3 &= R_3 \\ R_4 &= \lambda^2 R_4 \\ R_5 &= \lambda R_5 \\ R_6 &= \lambda R_6. \end{aligned}$$

   Since $\lambda \in \mathbb{R} - \{-1, 0, 1\}$, we must have $R_1 = R_2 = R_4 = R_5 = R_6 = 0$ and R_3 is free.

2. Case $\lambda = \pm 1$: Suppose $\lambda = \pm 1$. If $\lambda = -1$, then evaluating (17) and (18) at $\theta = 0$ gives us

$$\begin{aligned} R_5 &= -R_5 \\ R_6 &= -R_6. \end{aligned}$$

   Hence, we must have $R_5 = R_6 = 0$. Next suppose $\lambda = 1$. Then (17) and (18) become

$$\begin{aligned} R_5 &= \cosh\theta R_5 + \sinh\theta R_6 \\ R_6 &= \sinh\theta R_5 + \cosh\theta R_6. \end{aligned}$$

   If we multiply R_5 and R_6 by $\cosh\theta$ and $-\sinh\theta$ respectively, we see

$$\begin{aligned} \cosh\theta R_5 &= \cosh^2\theta R_5 + \cosh\theta \sinh\theta R_6 \\ -\sinh\theta R_6 &= -\sinh^2\theta R_5 - \cosh\theta \sinh\theta R_6. \end{aligned}$$

   If we add these two equations, we get $R_5 = (\cosh^2\theta - \sinh^2\theta) R_5 = \cosh\theta R_5 - \sinh\theta R_6$. But since $R_5 = \cosh\theta R_5 + \sinh\theta R_6$, we have

$$\cosh\theta R_5 + \sinh\theta R_6 = \cosh\theta R_5 - \sinh\theta R_6,$$

   or in other words,
$$2 \sinh\theta R_6 = 0.$$

   Since this equation must hold for all $\theta \in \mathbb{R}$, if $\theta \neq 0$, then $\sinh\theta \neq 0$ so we must have $R_6 = 0$. Substituting $R_6 = 0$ into (17), we have $R_5 = \cosh\theta R_5$. Again, this equation must hold for all $\theta \in \mathbb{R}$, so if $\theta \neq 0$, then $\cosh\theta \neq 1$ and hence $R_5 = 0$.

Next we will examine Equations (13),(14), and (16). Notice that if $\lambda = -1$ or if $\lambda = 1$, the equations are identical. Also since these equations must hold for all $\theta \in \mathbb{R}$, they must hold for when $\theta \neq 0$ as well, so for the sake of our calculations, suppose $\theta \neq 0$. If we add (13) and (14), we see that

$$R_1 + R_2 = (\cosh^2\theta + \sinh^2\theta)(R_1 + R_2) - 4\cosh\theta\sinh\theta R_4 = \cosh 2\theta(R_1 + R_2) - 2\sinh 2\theta R_4.$$

Hence,

$$(1 - \cosh 2\theta)(R_1 + R_2) = -2\sinh 2\theta R_4.$$

Since $\theta \neq 0$, $1 - \cosh 2\theta \neq 0$ and so

$$R_1 + R_2 = \frac{-2\sinh 2\theta}{1 - \cosh 2\theta} R_4.$$

We can simplify (16) as

$$\begin{aligned} R_4 &= -\cosh\theta\sinh\theta R_1 - \cosh\theta\sinh\theta R_2 + (\cosh^2\theta + \sinh^2\theta)R_4 \\ &= -\cosh\theta\sinh\theta(R_1 + R_2) + \cosh 2\theta R_4 \\ &= -\frac{1}{2}\sinh 2\theta(R_1 + R_2) + \cosh 2\theta R_4. \end{aligned} \tag{$\sharp$}$$

If we substitute $R_1 + R_2 = \frac{-2\sinh 2\theta}{1-\cosh 2\theta} R_4$ into ($\sharp$), we see

$$\begin{aligned} R_4 &= \left(\frac{-2\sinh 2\theta}{1 - \cosh 2\theta} R_4\right)\left(-\frac{1}{2}\sinh 2\theta\right) R_4 + \cosh 2\theta R_4 \\ &= \frac{\sinh^2 2\theta}{1 - \cosh 2\theta} R_4 + \cosh 2\theta R_4 \\ &= \frac{\cosh^2 2\theta - 1}{1 - \cosh 2\theta} R_4 + \cosh 2\theta R_4 \\ &= -(1 + \cosh 2\theta)R_4 + \cosh 2\theta R_4 \\ &= -R_4 - \cosh 2\theta R_4 + \cosh 2\theta R_4 = -R_4. \end{aligned}$$

Hence, $R_4 = -R_4$ and so $R_4 = 0$. We can plug in $R_4 = 0$ into $R_1 + R_2 = \cosh 2\theta(R_1 + R_2) - 2\sinh 2\theta R_4$ to see that

$$R_1 + R_2 = \cosh 2\theta(R_1 + R_2).$$

If $\theta \neq 0$, then $\cosh 2\theta \neq 1$ and so we must have $R_1 + R_2 = 0$, or in other words, $R_1 = -R_2$. So if $\lambda = \pm 1$, we have that $R_1 = -R_2$, R_3 is free, and $R_4 = R_5 = R_6 = 0$.

In all possible cases of $\lambda \in \mathbb{R} - \{0\}$, we have that $R_1 = -R_2$, R_3 is free, and $R_4 = R_5 = R_6 = 0$. Thus, the subspace of $\mathcal{A}(V)$ whose elements are preserved by a hyperbolic boost is the set of curvature tensors whose components are of the following form:

$$\begin{gathered} R_1 = -R_2 \\ R_3 \text{ is free} \\ R_4 = R_5 = R_6 = 0. \end{gathered}$$

□

Let $T \subseteq \mathcal{A}(V)$ be the subspace spanned by ACTs of the form above. By Claim 3.1, we have shown that any $R \in T$ is preserved by a hyperbolic boost, so $A_\theta \in G_R$ for all $\theta \in \mathbb{R}$. As we did in Section 2.1, for each possible $R \in T$ depending on $R_1, R_3 \in \mathbb{R}$, we will examine the Lie algebra structure of G_R. Notice that in Calculation 2.2, we calculated $A^* R_i = R_i$ where $R_4 = R_5 = R_6 = 0$ for some arbitrary $A \in G_R$. We disregarded $R_1 = R_2$ until the calculation was over, so we may use the results from Calculation 2.2.

So, let $R \in T$ and let $A \in G_R$. Also, let $A(t)$ be a path in G_R through I_3 such that $A(0) = I_3$. We must have that $A(t)^* R_i = R_i$ for all t. Then if we take the derivative of each side with respect to t and evaluate at $t = 0$, recall that (♣) says that

$$\left(\frac{\partial R_i}{\partial t}\right)_{|t=0} = \begin{pmatrix} 2R_1(x_1' + x_5') \\ 2R_2(x_1' + x_9') \\ 2R_3(x_5' + x_9') \\ R_2 x_6' + R_1 x_8' \\ R_3 x_3' + R_1 x_7' \\ R_3 x_2' + R_2 x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we can substitute $R_1 = -R_2$ into these equations to see that

$$\begin{pmatrix} 2R_1(x_1' + x_5') \\ -2R_1(x_1' + x_9') \\ 2R_3(x_5' + x_9') \\ -R_1 x_6' + R_1 x_8' \\ R_3 x_3' + R_1 x_7' \\ R_3 x_2' - R_1 x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For convenience, we label $R_1 = \gamma$ and $R_3 = \tau$. This gives us

$$\begin{pmatrix} 2\gamma(x_1' + x_5') \\ -2\gamma(x_1' + x_9') \\ 2\tau(x_5' + x_9') \\ \gamma(-x_6' + x_8') \\ \tau x_3' + \gamma x_7' \\ \tau x_2' - \gamma x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{♠}$$

We now examine all possible cases of $\gamma, \tau \in \mathbb{R}$.

1. Case $\gamma = \tau = 0$: Let $R \in T$ as above such that $\gamma = \tau = 0$. Then the equations in (♠) hold for all choices of x_j and hence each x_j is free for all $j = 1, \dots, 9$. Thus, $\dim G_R = 9$ and so $G_R = \mathrm{GL}(3, \mathbb{R})$. In this case, $R = 0$, the zero curvature tensor.

2. Case $\gamma = 0$ and $\tau \neq 0$: Let $R \in T$ such that $\gamma = 0$ and $\tau \neq 0$. Then the equations in (♠) become

$$\begin{pmatrix} 0 \\ 0 \\ 2\tau(x_5' + x_9') \\ 0 \\ \tau x_3' \\ \tau x_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

So we see that $x_5 = -x_9$, $x_2 = x_3 = 0$ and $x_1, x_4, x_6, x_7, x_8, x_9$ are free. Thus, $\dim G_R = 6$ and a basis $\beta_{\mathfrak{g}_R}$ for $\mathfrak{g}_R$ is

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

In this case, since $0 = \gamma = R_{1221} = R_{1331}$, we have that $e_1 \in \ker R$. Since R has non-trivial kernel, we can decompose R as $R = R_1 \oplus R_2$ and V as $V = V_1 \oplus V_2$, where $V_1 = \mathrm{span}\{e_1\}$ and $V_2 = \mathrm{span}\{e_2, e_3\}$ and each $R_i \in \mathcal{A}(V_i)$.

3. Case $\gamma \neq 0$ and $\tau = 0$: Let $R \in T$ such that $\gamma \neq 0$ and $\tau = 0$. Then the equations in (♠) become

$$\begin{pmatrix} 2\gamma(x_1' + x_5') \\ -2\gamma(x_1' + x_9') \\ 0 \\ \gamma(-x_6' + x_8') \\ \gamma x_7' \\ \gamma x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can see that $x_1 = -x_5 = -x_9$, $x_6 = x_8$, and $x_4 = x_7 = 0$. Thus, x_1, x_2, x_3, x_8 are free and so $\dim G_R = 4$. Then we have a basis $\beta_{\mathfrak{g}_R}$ of $\mathfrak{g}_R$ given by:

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

In this case, we have $R = R_{\phi_1} + R_{\phi_2}$ where $\phi_1, \phi_2 \in S^2(V^*)$ and

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\gamma \end{pmatrix}.$$

4. Case $\gamma \neq 0$ and $\tau \neq 0$: Let $R \in T$ such that $\gamma \neq 0$ and $\tau \neq 0$. Then the equations in (♠) become

$$\begin{pmatrix} 2\gamma(x_1' + x_5') \\ -2\gamma(x_1' + x_9') \\ 2\tau(x_5' + x_9') \\ \gamma(-x_6' + x_8') \\ \tau x_3' + \gamma x_7' \\ \tau x_2' - \gamma x_4' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we solve for each x_j, we see that $x_1 = x_5 = x_9 = 0$, $x_6 = x_8$, $x_3 = \frac{-\gamma}{\tau} x_7$, and $x_2 = \frac{\gamma}{\tau} x_4$. Thus, we have three free variables given by x_4, x_7, x_8. These then give us three basis vectors for basis $\beta_{\mathfrak{g}_R}$ of $\mathfrak{g}_R$, given by:

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma}{\tau} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{\gamma}{\tau} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

In this case, we have that $R = \operatorname{sgn}(-\tau) R_\phi$, where $\phi \in S^2(V^*)$ is given by

$$\phi = \begin{pmatrix} \operatorname{sgn}(-\tau)\gamma/\sqrt{|\tau|} & 0 & 0 \\ 0 & \sqrt{|\tau|} & 0 \\ 0 & 0 & -\sqrt{|\tau|} \end{pmatrix}.$$

So, we must have that $\dim G_R = 3$. Moreover, $G_R = O(1,2)$ or $G_R = O(2,1)$, depending on the signature of ϕ.

We have now classified all $R \in \mathcal{A}(V)$ in dimension 3 that are preserved by a hyperbolic boost. The possible dimensions of G_R are 9,6,4, and 3. In the case that $\dim G_R = 9$, we have that $G_R = \mathrm{GL}(3, \mathbb{R})$. In the case that $\dim G_R = 3$, we have either $G_R = O(1,2)$ or $G_R = O(2,1)$.

4 Model Spaces $\mathfrak{M}$ in dimension 3 preserved by Rotation Actions

We now focus our attention to model spaces $\mathfrak{M}$ of dimension 3. One notable feature about model spaces is that the vector space V is equipped with an inner product $\langle \cdot, \cdot \rangle$. Since an inner product is symmetric and bilinear, all of the information regarding an inner product can be deduced from applying the inner product to all combinations of basis vectors. In this section, we work over V of dimension 3, so let $\beta_V = \{e_1, e_2, e_3\}$ be a basis for V. We will also let H denote the $\langle \cdot, \cdot \rangle$ such that $H_{ij} = \langle e_i, e_j \rangle$. Up to the symmetries of H, there are six possible combinations of basis vectors to input into H, given by $H_{11}, H_{22}, H_{33}, H_{12}, H_{13}, H_{23}$. We also enumerate these as

$$\begin{aligned} H_1 &= H_{11}, \\ H_2 &= H_{22}, \\ H_3 &= H_{33}, \\ H_4 &= H_{12}, \\ H_5 &= H_{13}, \\ H_6 &= H_{23}. \end{aligned}$$

We will refer to the possible entries of H as above throughout the rest of the section.

4.1 Model Spaces $\mathfrak{M}$ preserved by $\mathbb{S}^1$

We will derive all possible model spaces $\mathfrak{M}$ over dimension 3 whose structure groups contain $\mathbb{S}^1$. We first start with a preliminary result.

PROPOSITION 4.1. *Define*

$$\mathbb{S}^1_{\pm 1} := \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \text{ and } \varepsilon = \pm 1 \right\}.$$

Then $\mathbb{S}^1_{\pm 1}$ is a group under multiplication of matrices.

Proof. Observe that $I_3 \in \mathbb{S}^1_{\pm 1}$ since I_3 is a matrix in $\mathbb{S}^1_{\pm 1}$ where $\theta = 0$ and $\varepsilon = 1$. Thus, $\mathbb{S}^1_{\pm 1}$ has an identity element. Since $\mathbb{S}^1_{\pm 1} \subseteq \mathrm{GL}(3, \mathbb{R})$ and it is known that multiplication of matrices in $\mathrm{GL}(3, \mathbb{R})$ is associative, we have that multiplication in $\mathbb{S}^1_{\pm 1}$ is also associative. Lastly, we must check for inverses. Let

$$X = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{pmatrix} \in \mathbb{S}^1_{\pm 1}.$$

Then consider the matrix

$$Y = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos(2\pi - x) & -\sin(2\pi - x) \\ 0 & \sin(2\pi - x) & \cos(2\pi - x) \end{pmatrix}.$$

Observe that $Y \in \mathbb{S}^1_{\pm 1}$ since if $x \in (0, 2\pi)$, $2\pi - x \in [0, 2\pi)$ as well. If $x = 0$, then replace $2\pi - x$ with 0, as the following calculations are equivalent. Then we can see that

$$\begin{aligned} XY &= \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos x & -\sin x \\ 0 & \sin x & \cos x \end{pmatrix} \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos(2\pi - x) & -\sin(2\pi - x) \\ 0 & \sin(2\pi - x) & \cos(2\pi - x) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(x + 2\pi - x) & -\sin(x + 2\pi - x) \\ 0 & \sin(x + 2\pi - x) & \cos(x + 2\pi - x) \end{pmatrix} = I_3. \end{aligned}$$

Hence, every $X \in \mathbb{S}^1_{\pm 1}$ has an inverse element $X^{-1} \in \mathbb{S}^1_{\pm 1}$ as well. Thus, $\mathbb{S}^1_{\pm 1}$ is a group. $\square$

Let $A_{\theta,\lambda}$ represent the $\mathbb{S}^1$ action as given in Section 2.1 and let $\mathfrak{M} = (V, \langle \cdot, \cdot \rangle, R) = (V, H, R)$ be a model space in dimension 3 such that $\mathbb{S}^1$ preserves $\mathfrak{M}$. In order for $\mathbb{S}^1$ to preserve $\mathfrak{M}$, $\mathbb{S}^1$ must preserve both R and H. In Section 2.1, we classified all $R \in \mathcal{A}(V)$ in dimension 3 that are preserved by $\mathbb{S}^1$. We use these results as well as new calculations of $A^*_{\theta,\lambda} H_i$ to determine which possible $\mathfrak{M}$ are preserved by $\mathbb{S}^1$. Since $\mathbb{S}^1$ preserves

$\mathfrak{M}$ by our supposition, we must have $A^*_{\theta,\lambda}H_i = H_i$ for all $i = 1, \ldots, 6$. If we compute $A^*_{\theta,\lambda}H_i = H_i$, we get the following equations:

$$H_1 = A^*_{\theta,\lambda}H_1 = \lambda^2 H_1 \tag{19}$$
$$H_2 = A^*_{\theta,\lambda}H_2 = \cos^2\theta H_2 + \sin^2\theta H_3 + 2\cos\theta\sin\theta H_6 \tag{20}$$
$$H_3 = A^*_{\theta,\lambda}H_3 = \sin^2\theta H_2 + \cos^2\theta H_3 - 2\cos\theta\sin\theta H_6 \tag{21}$$
$$H_4 = A^*_{\theta,\lambda}H_4 = \lambda\cos\theta H_4 + \lambda\sin\theta H_5 \tag{22}$$
$$H_5 = A^*_{\theta,\lambda}H_5 = -\lambda\sin\theta H_4 + \lambda\cos\theta H_5 \tag{23}$$
$$H_6 = A^*_{\theta,\lambda}H_6 = -\cos\theta\sin\theta H_2 + \cos\theta\sin\theta H_3 + (\cos^2\theta - \sin^2\theta)H_6, \tag{24}$$

all of which must hold for all $\theta \in [0, 2\pi)$ and fixed $\lambda \in \mathbb{R} - \{0\}$.

Claim 4.2. *Solving the equations above, we see H_1 is free and nonzero, $H_2 = H_3$ are free and nonzero, $H_4 = H_5 = H_6 = 0$, and we must have $\lambda = \pm 1$.*

Proof. To begin, we encode the data of H into the following array:

$$[H]_{ij} = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} = \begin{bmatrix} H_1 & H_4 & H_5 \\ H_4 & H_2 & H_6 \\ H_5 & H_6 & H_3 \end{bmatrix},$$

where the ij entry of $[H]_{ij}$ represents H_{ij}. We will fill in the entries of $[H]$ as we continue.

Notice that Equations (19)$-$(24) look nearly identical to Equations (1)$-$(6) in Section 2.1, which we have already solved. We will not repeat the calculations done in Claim 2.2; we instead focus on how they are identical up to a substitution and choice of λ for Equations (1)$-$(6).

We first examine Equations (22) and (23). If we replace H_4 with R_6 and H_5 with R_5, we see that (22) and (23) respectively become

$$R_6 = \lambda\cos\theta R_6 + \lambda\sin\theta R_5 = \lambda\sin\theta R_5 + \lambda\cos\theta R_6$$
$$R_5 = -\lambda\sin\theta R_6 + \lambda\cos\theta R_5 = \lambda\cos\theta R_5 - \lambda\sin\theta R_6.$$

Note that we did not actually equate $H_4 = R_6$ and $H_5 = R_5$, we simply just symbolically substituted R_6, R_5 in for H_4, H_5, respectively. In Claim 2.2, we showed that $R_5 = R_6 = 0$ for all cases of $\lambda \in \mathbb{R} - \{0\}$. Using this knowledge and realizing that Equations (22) and (23) are identical to (6) and (5) respectively up to the substitution we made, we can conclude that $H_4 = H_5 = 0$. We update these values in $[H]$ to see

$$[H] = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & H_6 \\ 0 & H_6 & H_3 \end{bmatrix}.$$

Next, if we symbolically substitute H_2 with R_1, H_3 with R_2, and H_6 with R_4, observe that Equations (20),(21),(24) are identical to Equations (1),(2),(4) in the case that $\lambda = \pm 1$ in Equations (1),(2),(4). In the case that $\lambda = \pm 1$ for Equations (1),(2),(4), we concluded that $R_4 = 0$ and $R_2 = R_3$. Hence, by substituting H_2 with R_1, H_3 with R_2, and H_6 with R_4, we see that $H_6 = 0$ and $H_2 = H_3$. Using this knowledge, we can update the entries of $[H]$:

$$[H] = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix}.$$

We hold off on substituting $H_2 = H_3$ in $[H]$ for now for the sake of a later calculation. Note that since H is an inner product, it must be nondegenerate. In other words, if H is nondegenerate, then $H(v, w) = 0$ for all $w \in V$ if and only if $v = \vec{0}$. If $H_2 = H_3 = 0$, then H would be degenerate since we would have $H(e_2, w) = 0$ for all $w \in V$, which cannot true since $e_2 \neq \vec{0}$ is a basis vector for V.

Finally, we examine Equation (19). Observe that if $\lambda \neq \pm 1$, then we conclude from Equation (19) that $H_1 = 0$. This would imply that $H(e_1, w) = 0$ for all $w \in V$. This cannot be the case since e_1 is a basis vector for V. Hence, we must have $\lambda = \pm 1$, which means that Equation (19) holds for any choice of H_1. □

So, if $\mathbb{S}^1$ preserves $\mathfrak{M}$, we have found that the inner product H of $\mathfrak{M}$ must be of the form above. We now examine $G_{\mathfrak{M}}$ as we did in Calculation 2.2 by looking at the Lie algebra structure of $\mathfrak{g}_{\mathfrak{M}}$.

Calculation 4.3. For some model space $\mathfrak{M}$ such that $R \in S$ and H is of the form given by Claim 4.2, we will calculate $A^*H_i = H_i$ for some arbitrary $A \in G_{\mathfrak{M}}$. We do not need to compute $A^*R_i = R_i$ since this calculation was already done in Calculation 2.3.

Firstly, let

$$A = \begin{pmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{pmatrix} \in G_{\mathfrak{M}}.$$

Since $A \in G_{\mathfrak{M}}$, we must have $A^*H_i = H_i$ for all $i = 1, \ldots, 6$. When we compute $A^*H_i = H_i$, we arrive at the following equations:

$$\begin{aligned} H_1 &= A^*H_1 = x_1^2H_1 + x_2^2H_2 + x_3^2H_3 \\ H_2 &= A^*H_2 = x_4^2H_1 + x_5^2H_2 + x_6^2H_3 \\ H_3 &= A^*H_3 = x_7^2H_1 + x_8^2H_2 + x_9^2H_3 \\ H_4 &= A^*H_4 = x_1x_4H_1 + x_2x_5H_2 + x_3x_6H_3 \\ H_5 &= A^*H_5 = x_1x_7H_1 + x_2x_8H_2 + x_3x_9H_3 \\ H_6 &= A^*H_6 = x_4x_7H_1 + x_5x_8H_2 + x_6x_9H_3. \end{aligned}$$

Since $G_{\mathfrak{M}}$ is a Lie group, we can find its dimension by finding the dimension of its Lie algebra $G_{\mathfrak{M}}$. So, let

$$A(t) = \begin{pmatrix} x_1(t) & x_4(t) & x_7(t) \\ x_2(t) & x_5(t) & x_8(t) \\ x_3(t) & x_6(t) & x_9(t) \end{pmatrix}$$

such that $A(0) = I_3$. As we did in Calculation 2.2, we can take the the the partial derivative of each H_i with respect to x_j to get the Jacobian matrix $\left(\frac{\partial H_i}{\partial x_j}\right)$. We then consider each x_j in $\left(\frac{\partial H_i}{\partial x_j}\right)$ as a function of t as described by the path $A(t)$ in $G_{\mathfrak{M}}$ such that $A(0) = I_3$. We can then evaluate each x_j at $t = 0$ and use the fact that $H_4 = H_5 = H_6 = 0$ to obtain the following modified Jacobian matrix $\left(\frac{\partial H_i}{\partial x_j}\right)_{|t=0}$:

$$\left(\frac{\partial H_i}{\partial x_j}\right)_{|t=0} = \begin{pmatrix} 2H_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2H_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2H_3 \\ 0 & H_2 & 0 & H_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_3 & 0 & H_2 & 0 \end{pmatrix}.$$

Now we can use the chain rule by multiplying the modified Jacobian matrix $\left(\frac{\partial H_i}{\partial x_j}\right)_{|t=0}$ by $\left(\frac{\partial x_j}{\partial t}\right)_{|t=0}$ to take the derivative of each H_i with respect to t and evaluating at $t = 0$. Since H_i is a real valued, constant, its derivative is 0. We will again let $x_j' = x_j'(0)$. Then we have

$$\left(\frac{\partial H_i}{\partial x_j}\right)_{|t=0} = \begin{pmatrix} 2H_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2H_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2H_3 \\ 0 & H_2 & 0 & H_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H_3 & 0 & 0 & 0 & H_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H_3 & 0 & H_2 & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ \vdots \\ \vdots \\ \vdots \\ x_9' \end{pmatrix} = \begin{pmatrix} 2H_1x_1' \\ 2H_2x_5' \\ 2H_3x_9' \\ H_2x_2' + H_1x_4' \\ H_3x_3' + H_1x_7' \\ H_3x_6' + H_2x_8' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{$\diamondsuit$}$$

This concludes Calculation 4.3. □

For the case of $\mathbb{S}^1$, we may now substitute $H_2 = H_3$ into ($\diamondsuit$) to get

$$\begin{pmatrix} 2H_1x_1' \\ 2H_2x_5' \\ 2H_2x_9' \\ H_2x_2' + H_1x_4' \\ H_2x_3' + H_1x_7' \\ H_2x_6' + H_2x_8' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{b}$$

So in order for $\mathbb{S}^1$ to preserve $\mathfrak{M}$, we must have that the Equations (b) and (a) in Section 2.1 hold. Observe that since H_1, H_2, H_3 are all nonzero, we must have that $x_1 = x_5 = x_9 = 0$. Also, we must have that $x_2 = -\frac{H_1}{H_2}x_4$, $x_3 = -\frac{H_1}{H_2}x_7$, and $x_6 = -x_8$. So, there are 3 free variables x_4, x_7, x_8 at the very most and hence $\dim G_{\mathfrak{M}} \leq 3$. As we did in Section 2.1, we will examine all possible $G_{\mathfrak{M}}$ that are preserved by $\mathbb{S}^1$ by examining all possibilities of $R_1 = R_2 = \alpha, R_3 = \beta \in \mathbb{R}$.

1. Case $\alpha = 0$ and $\beta = 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $A_{\theta,\pm 1}$, let $R \in S$ such that $\alpha = 0$ and $\beta = 0$. Then all of the equations in (a) always hold. We again must have that R is the zero curvature tensor. By the previous paragraph, since x_4, x_7, x_8 are free, $\dim G_{\mathfrak{M}} = 3$. The different possibilities of $G_{\mathfrak{M}}$ are $G_{\mathfrak{M}} = O(3)$, $G_{\mathfrak{M}} = O(2,1)$, $G_{\mathfrak{M}} = O(1,2)$, or $G_{\mathfrak{M}} = O(0,3)$, depending on the signature of H.

2. Case $\alpha = 0$ and $\beta \neq 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $A_{\theta,\pm 1}$, let $R \in S$ such that $\alpha = 0$ and $\beta \neq 0$. In Section 2.1, it was shown that the relationships among the variables for R in this case are $x_5 = -x_9$,$x_2 = x_3 = 0$, and $x_1, x_4, x_6, x_7, x_8, x_9$ are free. But since $\mathbb{S}^1$ preserves H as well, we have $x_1 = x_5 = x_9 = 0$, $x_2 = -\frac{H_1}{H_2}x_4$, $x_3 = -\frac{H_1}{H_2}x_7$, and $x_6 = -x_8$. Since $x_2 = x_3 = 0$, we must also have that $x_4 = x_7 = 0$. Also, since we have the relationship that $x_6 = -x_8$, we only have one free variable in this case, which is x_8. Thus, $\dim G_{\mathfrak{M}} = 1$ and a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ is

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

   Thus, $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathbb{S}^1_{\pm 1}$ as in Proposition 4.1.

3. Case $\alpha \neq 0, \beta = 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $A_{\theta,\pm 1}$, let $R \in S$ such that $\alpha \neq 0$ and $\beta = 0$. In Section 2.1, it was shown that the relationships among the variables for R in this case are $x_5 = x_9 = -x_1$, $x_6 = -x_8$, $x_4 = x_7 = 0$ and x_2, x_3 are free. But since $x_1 = x_5 = x_9 = 0$,$x_2 = -\frac{H_1}{H_2}x_4$, and $x_3 = -\frac{H_1}{H_2}x_7$, we must have $x_1 = x_5 = x_9 = 0$ as well as $x_2 = x_3 = 0$. Hence, we only have one free variable x_8. Thus, $\dim G_{\mathfrak{M}} = 1$ and a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ is

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

   Again, $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathbb{S}^1_{\pm 1}$ as in Proposition 4.1.

4. Case $\alpha \neq 0$ and $\beta \neq 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $A_{\theta,\pm 1}$, let $R \in S$ such that $\alpha \neq 0$ and $\beta \neq 0$. In Section 2.1, it was shown that the relationships among the variables for R in this case are $x_1 = x_5 = x_9 = 0$, $x_6 = -x_8$, $x_2 = -\frac{\alpha}{\beta}x_4$, and $x_3 = -\frac{\alpha}{\beta}x_7$. But, $x_2 = -\frac{H_1}{H_2}x_4$, $x_3 = -\frac{H_1}{H_2}x_7$ by our assumption that $\mathbb{S}^1$ preserves H. Hence, we have that $-\frac{\alpha}{\beta}x_4 = -\frac{H_1}{H_2}x_4$ and $-\frac{\alpha}{\beta}x_7 = -\frac{H_1}{H_2}x_7$, or in other words, $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right)x_4 = \left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right)x_7 = 0$. There are two cases here, depending on if $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) \neq 0$ or if $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) = 0$.

(a) Case $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) \neq 0$: If $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) \neq 0$, then we must have $x_4 = x_7 = 0$. In this case, there is only one free variable x_8. Therefore, $\dim G_{\mathfrak{M}} = 1$ and a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ is

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

Again, we have $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathbb{S}^1_{\pm 1}$.

(b) Case $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) = 0$: Suppose $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) = 0$. Observe that x_4 and x_7 are free, as well as x_8. Next, since $\left(\frac{H_1}{H_2} - \frac{\alpha}{\beta}\right) = 0$, we must have $\frac{H_1}{H_2} = \frac{\alpha}{\beta}$, or in other words, $H_1 = \frac{\alpha}{\beta} H_2$. We claim that R is actually $\pm R_H$, depending on the sign of β. To see this, let the array $[H]$ be

$$[H] = \begin{bmatrix} \operatorname{sgn}(\beta)\alpha/\sqrt{|\beta|} & 0 & 0 \\ 0 & \sqrt{|\beta|} & 0 \\ 0 & 0 & \sqrt{|\beta|} \end{bmatrix}$$

and let $R = \operatorname{sgn}(\beta) R_H$. The form of R_H is that given by Definition 2.1, where $H \in S^2(V^*)$. So,

$$R_1 = \operatorname{sgn}(\beta) H_1 \cdot H_2 = \operatorname{sgn}(\beta) \left(\operatorname{sgn}(\beta) \cdot \frac{\alpha}{\sqrt{|\beta|}} \sqrt{|\beta|} \right) = \alpha$$

$$R_2 = \operatorname{sgn}(\beta) H_1 \cdot H_3 = \operatorname{sgn}(\beta) \left(\operatorname{sgn}(\beta) \cdot \frac{\alpha}{\sqrt{|\beta|}} \sqrt{|\beta|} \right) = \alpha$$

$$R_3 = \operatorname{sgn}(\beta) H_2 \cdot H_3 = \operatorname{sgn}(\beta) \left(\sqrt{|\beta|}^2 \right) = \operatorname{sgn}(\beta)|\beta| = \beta$$

as desired. Since there are three free variables x_4, x_7, x_8, $\dim G_{\mathfrak{M}} = 3$ and a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ is

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -\frac{\alpha}{\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{\alpha}{\beta} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

Thus, $\dim G_{\mathfrak{M}} = 3$ and $G_{\mathfrak{M}} = O(1,2)$ or $G_{\mathfrak{M}} = O(3)$, depending on the signature of H.

This classifies $G_{\mathfrak{M}}$ of $\mathfrak{M}$ pereserved by $\mathbb{S}^1_{\pm 1}$ and hence $\mathbb{S}^1$ in the case that $\alpha \neq 0$ and $\beta \neq 0$ for $R \in S$.

This is now a complete classification of $\mathfrak{M}$ in dimension 3 such that $\mathbb{S}^1_{\pm 1}$ and hence $\mathbb{S}^1$ preserves $\mathfrak{M}$. We have found all possible dimensions of $G_{\mathfrak{M}}$, corresponding to 1 and 3. In the case that $\dim G_{\mathfrak{M}} = 1$, we have that $G_{\mathfrak{M}} = \mathbb{S}^1_{\pm 1}$. In the case that $\dim G_{\mathfrak{M}} = 3$, we have either $G_{\mathfrak{M}} = O(3)$ or $G_{\mathfrak{M}} = O(1,2)$.

4.2 Model Spaces $\mathfrak{M}$ preserved by a hyperbolic boost

Now we will study $\mathfrak{M}$ that are preserved by a hyperbolic boost. We start with a preliminary result.

Proposition 4.4. *Define*

$$\mathrm{HB} := \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cosh\theta & -\sinh\theta \\ 0 & -\sinh\theta & \cosh\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \text{ and } \varepsilon = \pm 1 \right\}.$$

Then HB *is a group under multiplication of matrices.*

Proof. Observe that $I_3 \in \mathrm{HB}$ since I_3 is a matrix in HB where $\theta = 0$ and $\varepsilon = 1$. Thus, HB has an identity element. Since $\mathrm{HB} \subseteq \mathrm{GL}(3,\mathbb{R})$ and it is known that multiplication of matrices in $\mathrm{GL}(3,\mathbb{R})$ is associative, we have that multiplication in HB is also associative. Lastly, we must check for inverses. Let

$$X = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cosh x & -\sinh x \\ 0 & -\sinh x & \cosh x \end{pmatrix} \in \mathrm{HB}.$$

Then consider the matrix

$$Y = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cosh(-x) & -\sinh(-x) \\ 0 & -\sinh(-x) & \cosh(-x) \end{pmatrix}.$$

Observe that $Y \in \mathrm{HB}$ since if $x \in \mathbb{R}$, $-x \in \mathbb{R}$ as well. Then we can see that

$$XY = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cosh x & -\sinh x \\ 0 & -\sinh x & \cosh x \end{pmatrix} \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cosh(-x) & -\sinh(-x) \\ 0 & -\sinh(-x) & \cosh(-x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(x-x) & -\sinh(x-x) \\ 0 & -\sinh(x-x) & \cosh(x-x) \end{pmatrix} = I_3.$$

Hence, every $X \in \mathrm{HB}$ has an inverse element $X^{-1} \in \mathrm{HB}$ as well. Thus, HB is a group. ☐

Let $B_{\theta,\lambda}$ be the same as that of Section 3. As in Section 4.1, we will calculate $B^*_\theta H_i = H_i$ for all i and determine what each H_i must be. So, suppose $\mathfrak{M}$ is preserved by a hyperbolic boost, or in other words, $B^*_{\theta,\lambda} R_i = R_i$ and $B^*_{\theta,\lambda} H_i = H_i$ for all i. In Section 3, we have found all $R \in \mathcal{A}(V)$ that are preserved by a hyperbolic boost, so we focus our attention to $B^*_{\theta,\lambda} H_i = H_i$. If we compute $B^*_{\theta,\lambda} H_i = H_i$, we get the following equations:

$$H_1 = B^*_{\theta,\lambda} H_1 = \lambda^2 H_1 \tag{25}$$
$$H_2 = B^*_{\theta,\lambda} H_2 = \cosh^2\theta H_2 + \sinh^2\theta H_3 - 2\cosh\theta\sinh\theta H_6 \tag{26}$$
$$H_3 = B^*_{\theta,\lambda} H_3 = \sinh^2\theta H_2 + \cosh^2\theta H_3 - 2\cosh\theta\sinh\theta H_6 \tag{27}$$
$$H_4 = B^*_{\theta,\lambda} H_4 = \lambda\cosh\theta H_4 - \lambda\sinh\theta H_5 \tag{28}$$
$$H_5 = B^*_{\theta,\lambda} H_5 = -\lambda\sinh\theta H_4 + \lambda\cosh\theta H_5 \tag{29}$$
$$H_6 = B^*_{\theta,\lambda} H_6 = -\cosh\theta\sinh\theta H_1 - \cosh\theta\sinh\theta H_2 + (\cosh^2\theta + \sinh^2\theta) H_6. \tag{30}$$

Claim 4.5. *We claim that H_1 is free and nonzero, $H_2 = -H_3$ are both nonzero, and $H_4 = H_5 = H_6 = 0$. In addition, we must have $\lambda = \pm 1$.*

Proof. We first examine Equations (28) and (29). Observe that if we add these two equations, we get

$$H_4 + H_5 = \lambda(\cosh\theta - \sinh\theta)(H_4 + H_5) = \lambda e^{-\theta}(H_4 + H_5).$$

In other words, we have $(\lambda e^{-\theta} - 1)(H_4 + H_5) = 0$. If $\lambda < 0$, $(\lambda e^{-\theta} - 1) \neq 0$ for all $\theta \in \mathbb{R}$ so we must have that $H_4 + H_5 = 0$. If $\lambda > 0$, when $\theta = \log(\lambda)$, we have that $(\lambda e^{-\theta} - 1) = (1 - 1) = 0$. But since $e^{-\theta} : \mathbb{R} \to \mathbb{R}_{>0}$ is injective, when $\theta \neq \log(\lambda)$, $(\lambda e^{-\theta} - 1) \neq 0$. But the equality $(\lambda e^{-\theta} - 1)(H_4 + H_5) = 0$ must hold for all $\theta \in \mathbb{R}$, so we must have $H_4 + H_5 = 0$. In all cases of $\lambda \in \mathbb{R} - \{0\}$, we have shown that it must be the case that $H_4 + H_5 = 0$, or in other words, $H_4 = -H_5$. We can substitute $H_4 = -H_5$ into (28) to see

$$H_4 = \lambda(\cosh\theta + \sinh\theta) H_4 = \lambda e^\theta H_4.$$

In other words, we must have $(\lambda e^\theta - 1) H_4 = 0$. If $\lambda < 0$, then $\lambda e^\theta - 1 < 0$ and so we must have that $H_4 = 0$. If $\lambda > 0$, since e^θ is injective, whenever $\theta \neq -\log(\lambda)$, we have that $\lambda e^\theta - 1 \neq 0$. But $(\lambda e^\theta - 1) H_4 = 0$ must hold for all $\theta \in \mathbb{R}$ so it must be the case that $H_4 = 0$. Since $H_4 = -H_5$, we see that $H_5 = 0$ as well.

Now we move on to examining Equations (26),(27), and (30). As we did in Claim 4.2, we will not directly solve these equations but rather point out that Equations (26),(27),(30) are identical to Equations (13),(14),(16) respectively, when $\lambda = \pm 1$ in Equations (13),(14),(16). We will identify H_2 with R_1, H_3 with R_2, and H_6 with R_4. In the case that $\lambda = \pm 1$ in Equations (13),(14),(16), we have shown that $R_1 = -R_2$ and $R_4 = 0$. Using this with the identification H_2 with R_1, H_3 with R_2, and H_6 with R_4, we conclude $H_2 = -H_3$ and $H_6 = 0$. As in Claim 4.2, we encode this information into the array $[H]$ to see that

$$[H] = \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix}.$$

Since H is an inner product, it must be nondegenerate. Observe that by equation (25) says that $H_1 = \lambda^2 H_1$. If $\lambda \neq \pm 1$, we must have that $H_1 = 0$ which cannot be true since if $H_1 = 0$, H would not be nondegenerate. Thus, we have that $\lambda = \pm 1$. ☐

Note 4.6. In Claim 4.5, we showed that if $B_{\theta,\lambda}$ preserves an inner product H, we must have $\lambda = \pm 1$. Hence, $B_{\theta,\lambda} \in \mathrm{HB}$ as in Proposition 4.4.

We now have described what what form an inner product H of $\mathfrak{M}$ must be in. As before, we use the fact that $G_{\mathfrak{M}}$ is a Lie group to list the possible dimensions of $G_{\mathfrak{M}}$ that are preserved by a hyperbolic boost. So, let $\mathfrak{M}$ be a model space preserved by a hyperbolic boost and consider a path $A(t)$ through I_3 in $G_{\mathfrak{M}}$ such that $A(0) = I_3$. We then repeat the process as done Calculation 2.2. Observe that at the end of Calculation 2.2, ($\diamondsuit$) holds for any path $A(t)$ through any $G_{\mathfrak{M}}$ such that $H_4 = H_5 = H_6 = 0$. Since that is the case here, we will reuse these results. Recall that ($\diamondsuit$) says that

$$\begin{pmatrix} 2H_1x_1' \\ 2H_2x_5' \\ 2H_3x_9' \\ H_2x_2' + H_1x_4' \\ H_3x_3' + H_1x_7' \\ H_3x_6' + H_2x_8' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we substitute $H_2 = -H_3$, we see that

$$\begin{pmatrix} 2H_1x_1' \\ 2H_2x_5' \\ -2H_2x_9' \\ H_2x_2' + H_1x_4' \\ -H_2x_3' + H_1x_7' \\ -H_2x_6' + H_2x_8' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{$\heartsuit$}$$

Thus, we must have that $x_1 = x_5 = x_9 = 0$, $x_2 = -\frac{H_1}{H_2}x_4$, $x_3 = \frac{H_1}{H_2}x_7$, and $x_6 = x_8$, so x_4, x_7, x_8 are free. As we did in Section 4.1, we examine all possible $G_{\mathfrak{M}}$ that are preserved by a hyperbolic boost by examining all possibilities of curvature entries of R $R_1 = -R_2 = \gamma, R_3 = \tau \in \mathbb{R}$ in addition to all possible inner product entries of H.

1. Case $\gamma = \tau = 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $B_{\theta,\pm 1}$, let $R \in T$ such that $\gamma = \tau = 0$. Then the equations in ($\spadesuit$) always hold for any choice of x_j and R is the zero curvature tensor. By the previous paragraph, x_4, x_7, x_8 are free, so $\dim G_{\mathfrak{M}} = 3$ and $G_{\mathfrak{M}} = O(2, 1)$ or $G_{\mathfrak{M}} = O(1, 2)$, depending on the signature of H.

2. Case $\gamma = 0$ and $\tau \neq 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $B_{\theta,\pm 1}$, let $R \in T$ such that $\gamma = 0$ and $\tau \neq 0$. By Case 2 in Section 3, we have that $x_5 = -x_9$, $x_2 = x_3 = 0$ and $x_1, x_4, x_6, x_7, x_8, x_9$ are free. But by the equations in ($\heartsuit$), we have $x_1 = x_5 = x_9 = 0$, $x_2 = \frac{-H_1}{H_2}x_4$, $x_3 = \frac{H_1}{H_2}x_7$, and $x_6 = x_8$. So, we must have $x_1 = x_5 = x_9 = 0$, $x_2 = x_3 = x_4 = x_7 = 0$, and $x_6 = x_8$, which means that there is one free variable x_8. This gives us a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$, given by

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

   Thus, $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathrm{HB}$ as in Proposition 4.4.

3. Case $\gamma \neq 0$ and $\tau = 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $B_{\theta,\pm 1}$, let $R \in T$ such that $\gamma \neq 0$ and $\tau = 0$. By Case 3 in Section 3, we have that $x_1 = -x_5 = -x_9$, $x_6 = x_8$, and $x_4 = x_7 = 0$. But we must also have that $x_1 = x_5 = x_9 = 0$, $x_2 = \frac{-H_1}{H_2}x_4$, and $x_3 = \frac{H_1}{H_2}x_7$ since $B_{\theta,\pm 1}$ preserves $\mathfrak{M}$. So in total, we can see that $x_1 = x_5 = x_9 = 0$, $x_2 = x_3 = x_4 = x_7 = 0$, and $x_6 = x_8$. Hence, we have one free variable x_8. This gives us a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$, given by

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

   So, $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathrm{HB}$.

4. Case $\gamma \neq 0$ and $\tau \neq 0$: For model space $\mathfrak{M} = (V, H, R)$ preserved by $B_{\theta,\pm 1}$, let $R \in T$ such that $\gamma \neq 0$ and $\tau \neq 0$. Then we must have $x_1 = x_5 = x_9 = 0$, $x_6 = x_8$, $x_3 = -\frac{\gamma}{\tau} x_7$, and $x_2 = \frac{\gamma}{\tau} x_4$. Since $B_{\theta,\pm 1}$ also preserves $\mathfrak{M}$, we must also have $x_2 = -\frac{H_1}{H_2} x_4$ and $x_3 = \frac{H_1}{H_2} x_7$. We then can see that $\frac{\gamma}{\tau} x_4 = -\frac{H_1}{H_2} x_4$ and $-\frac{\gamma}{\tau} x_7 = \frac{H_1}{H_2} x_7$, or in other words,

$$\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) x_4 = \left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) x_7 = 0.$$

We then have two cases, depending on if $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) \neq 0$ or if $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) = 0$.

(a) Case $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) \neq 0$: Suppose $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) \neq 0$. Then we must have $x_4 = x_7 = 0$ which implies that $x_2 = x_3 = 0$. In this case, we only have one free variable x_8. Therefore, $\dim G_{\mathfrak{M}} = 1$ and a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ is

$$\beta_{\mathfrak{g}_{\mathfrak{M}}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Again, we have that $\dim G_{\mathfrak{M}} = 1$ and $G_{\mathfrak{M}} = \mathrm{HB}$.

(b) Case $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) = 0$: Suppose $\left(\frac{\gamma}{\tau} + \frac{H_1}{H_2}\right) = 0$. We claim that $R = \operatorname{sgn}(-\tau) R_H$. To see this, let the array $[H]$ be

$$[H] = \begin{bmatrix} \operatorname{sgn}(-\tau)\gamma/\sqrt{|\tau|} & 0 & 0 \\ 0 & \sqrt{|\tau|} & 0 \\ 0 & 0 & \sqrt{|\tau|} \end{bmatrix}$$

and let $R = \operatorname{sgn}(\beta) R_H$. Then we see that

$$\begin{aligned} R_1 &= \operatorname{sgn}(-\tau) H_1 \cdot H_2 = \operatorname{sgn}(-\tau) \left(\operatorname{sgn}(\tau) \cdot \frac{\gamma}{\sqrt{|\tau|}} \sqrt{|\tau|} \right) = \gamma, \\ R_2 &= \operatorname{sgn}(-\tau) H_1 \cdot H_3 = \operatorname{sgn}(-\tau) \left(\operatorname{sgn}(-\tau) \cdot \frac{\gamma}{\sqrt{|\tau|}} \cdot -\sqrt{|\tau|} \right) = -\gamma, \\ R_3 &= \operatorname{sgn}(-\tau) H_2 \cdot H_3 = \operatorname{sgn}(-\tau) \left(-\sqrt{|\tau|}^2 \right) = -\operatorname{sgn}(-\tau)|\tau| = \tau. \end{aligned}$$

In this case, we have three free variables x_4, x_7, x_8, so $\dim G_{\mathfrak{M}} = 3$ and we have a basis $\beta_{\mathfrak{g}_{\mathfrak{M}}}$ for $\mathfrak{g}_{\mathfrak{M}}$ given by

$$\beta_{\mathfrak{g}_R} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma}{\tau} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{\gamma}{\tau} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

In this case, since $R = R_\phi$ for some $\phi \in S^2(V^*)$, we have that $G_{R_\phi} = G_\phi$. So, we must have $G_R = O(1,2)$ or $G_R = (2,1)$, depending on the signature of ϕ.

This classifies $G_{\mathfrak{M}}$ of $\mathfrak{M}$ pereserved by a hyperbolic boost in the case that $\gamma \neq 0$ and $\tau \neq 0$ for $R \in T$.

In conclusion, we have provided a complete classification of the possible dimensions of $G_{\mathfrak{M}}$ for some model space $\mathfrak{M}$ that is preserved by a hyperbolic boost. There are two possible dimensions of $G_{\mathfrak{M}}$: $\dim G_{\mathfrak{M}} = 3$ and $\dim G_{\mathfrak{M}} = 1$. In the case that $\dim G_{\mathfrak{M}} = 1$, it is shown that $G_{\mathfrak{M}} = \mathrm{HB}$ as defined in Proposition 4.4. In one particular case of $\dim G_{\mathfrak{M}} = 3$, we have that $G_{\mathfrak{M}} = O(2,1)$ or $G_{\mathfrak{M}} = O(1,2)$.

5 Future Work

We provide some open questions for future work related to this paper.

1. We have shown that there are some curvature tensors $R = \pm R_{\phi_1} \pm R_{\phi_2}$ for $\phi_1, \phi_2 \in S^2(V^*)$ with a 4 dimensional structure group. Such an R cannot be expressed as a single R_ϕ. Is there a method of determining if a given R is of the form $R = R_{\phi_1} + R_{\phi_2}$ by looking at the dimension of G_R?

2. This paper focused on rotation actions in dimension 3. What would change (or stay the same) about these methods if we studied rotation actions in dimension $n > 3$?

3. One can think about the set of unit quaternions in $\mathbb{R}^4$ and realize this set as subset of $\mathrm{SL}(4, \mathbb{R})$. Is it possible that the set of unit quaternions is a structure group of some ACT R or model space $\mathfrak{M}$ in dimension 4? The quaternion group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ is a subset of the unit quaternions where each $\pm i, \pm j, \pm k$ have order 4. In this paper we have shown that in dimension 3, if an element of a structure group has finite order, it must contain all rotations $\mathbb{S}^1$. Does this result hold for higher dimensions and hence Q_8 cannot be a structure group?

4. Suppose G is some structure group. Since this structure group G is a Lie group, it is also a manifold. It is known that we can calculate the curvature of a given Lie group. Let R be an ACT of G. What is G_R? In other words, what is the structure group of an ACT R that measures the curvature of a structure group G?

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