All Links are Sublinks of Cuboctahedral Arithmetic Links

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Abstract

In this paper we prove that all links are sublinks of cuboctahedral links. We begin with with a cuboctahedral nested link and create a method to find any closed braid using Dehn twists and/or fillings. We then study the cusps of the crossing circles used and properties when doing Dehn fillings. Finally other possibilities in different parent links are considered.

1 Introduction: FALs and Nested Links

First, we will introduce links and some classifications, then go into proving our theorem and looking

To begin, we will look at some of the terminology of knot theory used in this paper:

- A knot is a closed curve in three dimensions
- A link is composed of intertwined knots (Every knot is a links with one component)
- A hyperbolic link is one whose complement is composed of ideal hyperbolic polyhedra
- A cuboctahedral link is one whose complement is constructed of ideal cuboctahedra

Fully Augmented Links(FALs) and Nested Links are a subclass of hyperbolic links. To augment a link diagram, we put a unknot called a crossing circle around each of the twist regions and undo full twists. Doing so creates disks in the complement of the links enclosed by the crossing circles which in FALs are twice punctured by knot strands. This is due to the fact that twist regions can have only two strands in FALs.

Nested links are much like FALs but twist regions can have more than two strands, and crossing circles can be nested inside each other, so long as each still encircles a disk, twice punctured either by knot strands or other crossing circles.

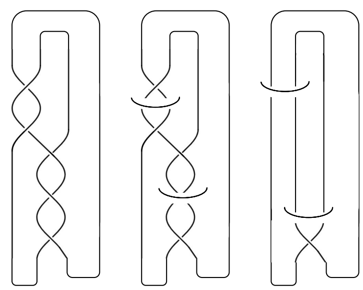

Figure 1: The Augmentation of a Link

In a paper by Mark D. Baker [1], he proves that all links are sublinks of arithmetic links, and uses an octahedral nested parent link, therefore also proving that all links are sublinks of octahedral links. In this paper, we start with the nested cuboctahedral link shows below and prove that all links are sublinks of cuboctahedral links.

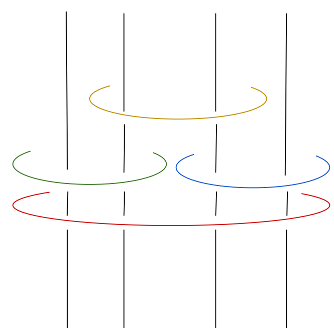

Figure 2: Our Nested Cuboctahedral Parent Link

2 Cell Decomposition and Circle Packing

Now that we have a bit of background about cuboctahedral links, FALs and nest links, we will discuss the cell decomposition and circle packing, which, when given hyperbolic links, are a way to get a 2-dimensional blueprint for what the ideal hyperbolic polyhedra of the complement look like. To find the cell decomposition, we:

1. Lay the knot circles flat upon the plane of projection and cut along this

plane, cutting the link and complement into two identical halves, P^+ and P^-.

2. Take each of the now half disks that the crossing circles enclose and do a butterfly slice along those, then flatten them out so the crossing circle's knot component goes through the middle.

3. Reduce each of the crossing circle and knot circle strands to ideal points

4. To get the circle packing, we just bend what we have into shape so that each of the faces from the plane of projection are circles.

5. We can then throw any of the ideal points, or points of tangency, to infinity to see what P^+ looks like in the fundamental region if we were looking at it from above, and to get the full complement we glue an exact copy of it, or P^-, along one of the vertical faces from the plane of projection. The result is the fundamental region of the link complement.

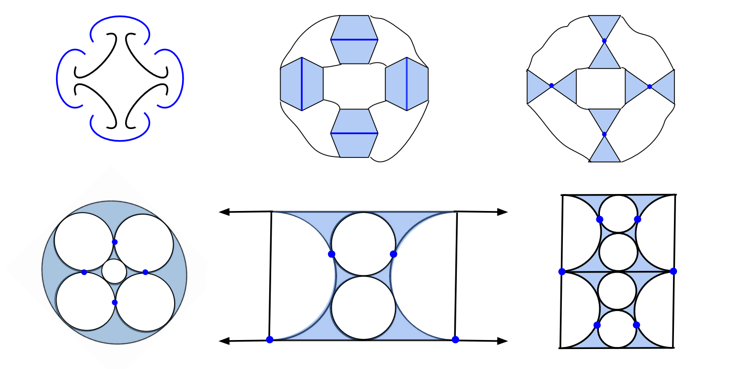

Figure 3: Cell Decomposition and Circle Packing of P_4

Now that we have the fundamental region, we can either shrink wrap it to the corresponding manifold in 3-space or study its 2-dimensional representation. We will look at both later on in this paper.

We also must note, however, that when we find the cell decomposition of a nested link, because one crossing circle lies inside the other, when the butterfly cut is performed on the outer crossing circle, the other one is cut in two. This means that there will be two ideal point corresponding to the inner crossing circle in the fundamental region, but each only represents one quarter and not one half of a crossing circle. An example of the butterfly cut in this case is shown in Figure 4, and the circle packing of our parent link is shown in Figure 5.

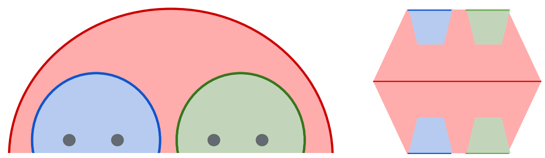

Figure 4: Nested Butterfly Cut

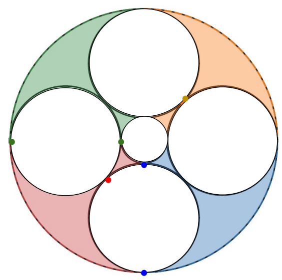

Figure 5: The Circle Packing of our Parent Link

As can be seen, it is similar to the circle packing of P_4, different only in its coloring and placement of ideal points corresponding to crossing circles. This means that their complements are homeomorphic.

In Proposition 6.4 by Chesebro, Deblois, and Wilton [2] they state that any cuboctahedral link of finite volume is arithmetic, and thus, our link is arithmetic.

3 $2r$-fold covers

In order to create our braids, use 2r fold cover to get sufficient strands to be in our braid and crossing circles to create each twist region while also maintaining its arithmeticity and cuboctahedral properties.To find a 2 fold cover of a link over a knot circle, we first arrange it so all the crossing circles sit horizontally, either nested or stacked, and each knot circle is a vertical line going through crossing circles then looping back around the outside and connecting to itself. This form looks much like the closure of a braid with all its crossings removed. Since each strand connects to itself we draw them cut off, as shown below. To find the 2 fold cover, we:

1. Take a knot circle throw a point to ∞ so its disk expands into a half-plane.

2. Do a half slice and flatten out the half-plane into a plane. All the crossing circles going through this knot circle will have 2 points in which they intersect this plane.

3. Take a copy of this arrangement and rotate it around the open knot circle, so that the planes face eachother and there is nothing between them.

4. Glue the planes together so all the crossing circles that were sliced open glue to their counterparts in the rotation.

5. Take the point back from ∞ to recover the knot circle.

Below is the process for finding the 2 fold cover of our cuboctahedral parent link.

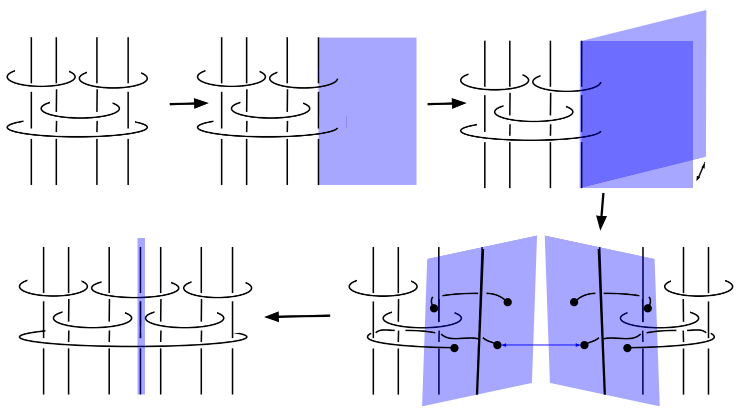

Figure 6: Finding a 2-fold cover

If we were to do the same thing again, we would end up with the 2-fold cover of the 2-fold cover, or, the 4-fold cover. We can keep taking the 2 fold cover of our covering spaces, doubling every time and finding a pattern for 2^r fold covers. We can also do the same process but instead of taking a copy of our link and gluing it plane-to-plane, we will take a 2-fold cover of our parent link and glue it to whatever link we have, thus making a pattern of $2r$-fold covers. We can also do the same thing and take $2r$-fold covers around a crossing circles, resulting in copies of this link going downwards.

When we do the 2r-fold cover, we are gluing together cuboctahedra in the complement and thus our link remains cuboctahedral, Proposition 6.4 [2] still works, and this does not change the arithmetic properties of our links.

4 2^r Dehn Twists

To do an m-integer Dehn twist, we do a half slice on the disk of a crossing circle, then blow it up into a sphere which is separated along the equator by the crossing circle. We then rotate the top half by $2m\pi$ radians and glue the disk back down. This creates m full twists for the strands going through this crossing circle. We can also do $m/2$, or half integer Dehn twists, which instead give m crossings.

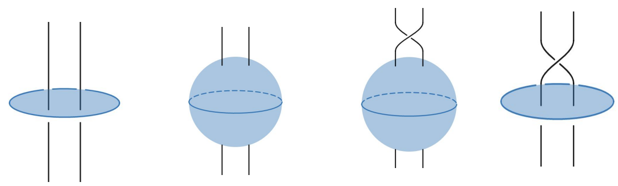

Figure 7: A 1/2 Dehn Twist

Because doing Dehn twists does not change the fact that our link is cuboctahedral, it only changes the gluing of these cuboctahedra, Proposition 6.4 [2] still holds and the link remains arithmetic.

5 Braids

Alexander's Theorem states that any knot or link can be represented as a closed braid. Thus, if we can find a way to make any closed braid as a sublink of cuboctahedral arithmetic links, we have proved that any knot or link is a sublink of cuboctahedral arithmetic links but first, we must introduce braids. To make a braid, we start with n strands beginning parallel traveling in the same direction. We then can cross any two consecutive strands however many times we want, as long as every strand is constantly traveling downward and does not loop back up. Doing so can be described by operations, $\alpha_1, \alpha_2, ..., \alpha_{n-1}$, where α_i is the crossing of the i^{th} over the $(i+1)^{th}$ strand. Doing any of these operations multiple times in a row is denoted α_i^p, where p is the number of times α_i was performed, which is also the number of crossings in this twist region. Doing any of these operations backwards, or crossing the $(i+1)^{th}$ strand over the i^{th} strand is denoted α_i^{-1}. An example of a braid is shown below on the left of Figure 8.

The closure of a braid takes each strand from the top and connects it with the corresponding strand in its place on the bottom. The i^{th} strand on the top connects with the i^{th} strand on the bottom. As previously stated, Alexander's theorem shows that we can represent any knot or link in this form.

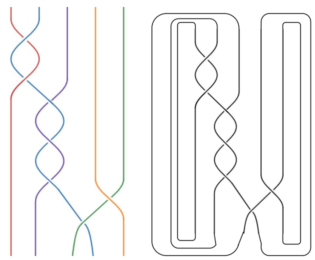

Figure 8: The $\alpha_1^{-2}\alpha_2^3\alpha_4\alpha_3^{-1}$ braid and its closure

6 Building Any Closed Braids

Now that we have sufficient background, we will illustrate the process of finding any closed braid and thus, any link, using the tools we have talked about. To build a braid from our link, we look at n, the number of strands, and t the number of twist regions in our braid, and take the $2\lceil n/4\rceil$-fold cover over the knot circle and $2\lceil n/2\rceil$-fold cover over the crossing circle. Then do a pattern of skipping one strand and highlighting the next two as the strands of our braid, until we have n of then. These will be the strands we use in our braid. Then, starting from the top, we look at the strands used in our first twist region and find the crossing circle that encircles them in our covering space and as many half-intereger Dehn twists as needed to replicate their crossing. We do the same for every twist region, making sure to twist them in a crossing circle below the prior twist region, so they are in order and our braid is replicated.

To better demonstrate how we can build a closed braid from covering spaces of our link we will show how to create the closed braid in Figure 1 using Dehn twists. This braid has 5 strands and 4 twist regions and thus, we will need 4 and 4 fold covers over knot and crossing circles respectively. To create this braid, we skip the first strand and highlight the next two, and repeat until we have 5 highlighted strands. These will be the strands in the braid we are trying to form. Then starting at the left, we do Dehn twists to cross strands in the order of $\alpha_1^{-2}\alpha_2^3\alpha_4\alpha_3^{-1}$, and thus, our braid is now a sublink of this cuboctahedral link.

7 Dehn Fillings and Cusps

Instead of Dehn twists, we could also do Dehn fillings in crossing circles, which will, not only twist the strands going through it but also remove the crossing circle. To do a Dehn filling, we will first look at the cusp of a crossing circle. The cusp of a knot component is its neighborhood in the complement, which, in the case of crossing circles is the torus neighborhood around a crossing circle. We can fill the cusp boundary with another torus by gluing meridian to meridian.

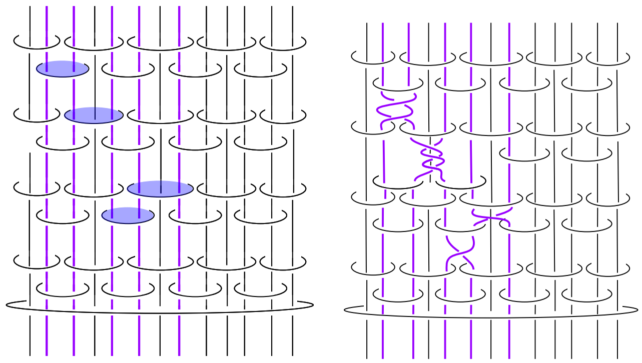

Figure 9: Creating the $\alpha_1^{-2}\alpha_2^3\alpha_4\alpha_3^{-1}$ braid

Since this solid torus is in the complement, the crossing circle inside it is no longer there. We can think of this a bit like a jelly doughnut, filling the torus of the doughnut with a torus of jelly if the doughnut and jelly were equal thickness. This is easy to visualize in Figure 10, where the meridians have the same shape and could easily glue to each other.

Figure 10: Gluing a Trivial Meridian to Meridian on a Cusp

What if we try to glue it to different meridional slope on the torus? Such as

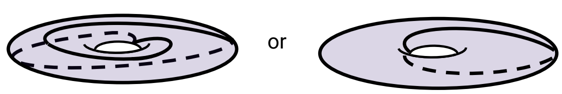

Figure 11: Other Slopes on the Torus

If we take the slop on the right of Figure 11, we can do one Dehn twist, which will twist the top half the torus, and end up with the torus in Figure 12, which, we can now move the point shown and get a slope that looks like one from Figure 10, so we can easily glue it now. Since we did a Dehn twists however, any strands going through this crossing circle will now be twisted.

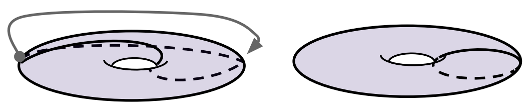

Figure 12:

To do a $1/w$ Dehn filling, we fill the torus along the slope of 1 meridian and w longitudes in the rectangle that the torus flattens to, and this will do w full twists on the strands inside. In Figure 13, we show how the slopes on meridian can be visualized on rectangles.

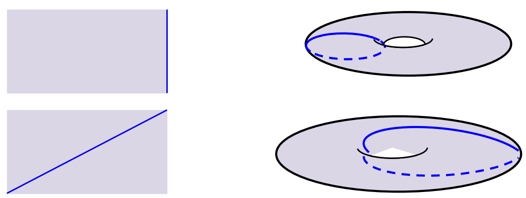

Figure 13:

8 Cusp shapes on our links

We will now look at the shape of our cusps on the crossing circles. In order to do so, we find where the cusp lies in relation to other regions of the link on the fundamental region of the ideal cuboctahedron. If we then set the meridian of one of the cusps to a certain value, we can get the size of the rest of them in this model.

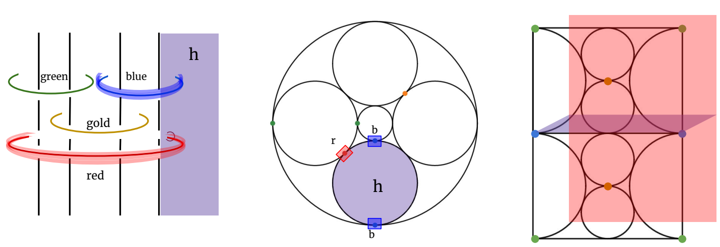

Figure 14: Finding the Size of the First Cusp

To begin, we looked at the highlighted region, h, on the right side of the link, and the two cusps that are highlighted. I first found the shape of the seemingly largest crossing circle at the bottom of our link diagram. We can see that the intersection here is with the meridian of this cusp and h, which is in the complement. When we find the circle packing of our link, the cusps become squares around ideal points and h becomes the circle on the bottom. When we throw the point of the crossing circle whose cusp we are trying to find to infinity, the cusp becomes a horizontal plane and h becomes a vertical plane. Since we know their intersection to be the length of the meridian, we know this length in the fundamental region to be the meridian the other side must be the longitude. We must remember that when we find the circle packing and throw a point to infinity, we have only found the top half, or P^+ in the complement, so we have to glue a copy of it, or P^- to get the full cusp. Now, we set the longitude of the red cusp to be 2 and we find that the meridian equals $\sqrt{2}$. We can either repeat this process with the rest of the cusps, keeping in mind that we have already set one value so we need only find the others, or since we know the fundamental region will look the same no matter which point we throw to infinity, we can make observations about the gluing of our manifold. Specifically, we recall that since the seemingly largest red crossing circle and cusp was split in half when we started to process of finding the fundamental region, this made the longitude twice as much as it appeared if we had just thrown a point in the circle packing to infinity. If we look at the blue crossing circle and cusp on the top right, we notice that its not only split in half once by the projection plane but once again by the red cusp, which it is nested inside. For this reason, there are two squares that correspond the the cusp of this crossing circle when we look at the circle packing and why we would, not only need to glue a P^- to our P^+ after throwing one of these ideal points to infinity but also glue another entire fundamental region. This results in a meridian that is double the size of the red cusp's meridian.

Once we do this for all the cusps, we find that in our original link, the gold cusp has the same meridian of $\sqrt{2}$ and longitude of 2 and the red (seemingly but not actually largest). The blue and green crossing circles which are nested alongside each other in the red crossing circle both have meridians of $2\sqrt{2}$ and longitude of 2.

Recall that when we take the 2 fold cover, some of the crossing circles are glued to copies of themselves, so the same happens to their cusp and the longitude doubles. This can also be seen in the fundamental region as doing this would glue on another cuboctahedron and the edge of the rectangular area corresponding to the meridian will be twice as long. The doubling of the meridian only happens once in the row of blue and green crossing circles because of the pattern that the 2r-fold cover follows, so the cusps in this row that do not have the shape previously stated have longitudes of 4 and meridians of $2\sqrt{2}$.

When we take the 2-fold cover, we also notice that the longitude of the red cusp doubles, but since we don't do Dehn filling on this crossing circle, the cusp shape is not important in this study.

9 Theorem 5.1 (Futer, Purcell, Kalfagianni)

[3] If M is a hyperbolic manifold, and $s_1, ..., s_k$ are slopes on cusps of M with minimum length l_{min} at least 2π, then the Dehn filled manifold $M(s_1, ..., s_k)$ is hyperbolic with volume bounded below by

$$vol(M(s_1, ..., s_k)) \geq (1 - (2\pi/l_{min})^2))^{3/2} vol(M)$$

When we do Dehn fillings, because we are removing a crossing circle, the volume of the link decreases but this theorem puts a lower bound on how much it can decrease by. We will now look at the volume of our link and how it changes with Dehn fillings but first, we must take into account the sentence in this theorem that states "slopes on cusps of M with minimum length l_{min} at least 2π". This refers to a theorem that states that the slope length on the cusp that we fill must be at least 2π in order for our new link to be hyperbolic. Using the measurements of our cusp shapes and the Pythagorean theorem, we find the minimum number of longitudes needed to get this length. For the gold cusp, we need at least 4 longitudes, and for the larger crossing circles that appear in the 2-fold cover in the row of green and blue, we need at least 2 longitudes. Notice that we do not calculate the minimum number of longitudes needed for the green, blue, or red crossing circles, this is because we do not actually use them in our process of creating any braid. This essentially means that in our process of creating braids, if we use Dehn filling instead of Dehn twists, these are the minimum slope length we must fill our cusps along for our new link to be hyperbolic.

Now, using Theorem 5.1, when we plug in our calculations, we get a lower bound for the volume to be

$$vol(M(s_1, ..., s_k)) \geq 0.3035699356 r(v_{cub})$$

when we do Dehn fillings only on the crossing circles in the blue and green row, and

$$vol(M(s_1, ..., s_k)) \geq 0.2547318474 r(v_{cub})$$

when we do Dehn fillings on any combination of the crossing circles, where we have taken the r-fold cover of our parent link since the number r is the amount of cuboctahedrons glued together and v_{cub} is the volume of our ideal cuboctahedron.

10 Future Research

Because our parent link is a nested link, the cusp on two of the crossing circles has a meridian that is twice as large as the other two crossing circles. Additionally, when we take the 2-fold cover, the longitude of some of the cusps doubles. Some future questions we could look into are:

- Is there a way to form a different cuboctahedral nested link such that some of the cusps have meridians 3 or 4 times as large as ours
- Is there a different series of Dehn fillings that excludes the crossing circles with smaller meridians, therefore reducing the minimum number of longitudes needed to get a length of 2π
- What can we say about the resulting manifolds in Lens spaces if we used 1 longitude and p meridians in our Dehn Fillings
- Can we classify different 3-manifolds we would get by using p meridians and q longitudes

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