DECOMPOSABILITY AND LINEAR INDEPENDENCE OF CANONICAL ALGEBRAIC CURVATURE TENSORS EXPRESSED THROUGH CHAIN COMPLEXES OVER PSEUDO-EUCLIDEAN VECTOR SPACES

THOMAS ZAMROZ

Abstract. We expand previous work on linear dependence of canonical algebraic curvature tensors constructed from symmetric and skew-symmetric linear transformations by describing them combintorially through directed graph chain complexes. Furthermore, we extend this combinatorial approach to tensor decomposability while formulating a new characterization for decomposability and introducing a weaker, more expansive definiton of k-decomposability.

1. Introduction

Algebraic curvatures tensors are mathematical objects that are pivotal to the study of Riemannian and pseudo-Riemannian maifolds. While there are many geometric implications that can be derived from their behavior, they have become a significant object of study in their own right.

To introduce where the idea of an algebraic curvature tensor arose we start with an n-dimensioal *Riemannian manifold* (M, g) where M is a real, smooth manifold and g_p is an *metric*, or *inner product*, on the *tangent vector space* T_pM of M at a point p. Whether or not this inner product is positive definite classifies M as either Riemannian or *pseudo-Riemannian*. Regardless, we can measure the curvature of this manifold at a point p through an *algebraic curvature tensor* which is a multilinear map $R_p : (T_pM)^4 \to \mathbb{R}$. Moreover, the triple (T_pM, g_p, R_p) is a *model space*.

The focus of this research is to further understand the behavior of algebraic curvature tensors and how they interact with each other.

1.1. Algebraic Curvature Tensors.

Throughout the paper, unless otherwise stated, it is assumed that V is a finite dimensional real vector space with $\dim(V) = n$. That being said, we begin with a definition.

Definition 1.1. Let $R \in \bigotimes^4(V^*)$. The multilinear mapping $R : V^4 \to \mathbb{R}$ is an *algebraic curvature tensor* on V if

(1) $R(x, y, z, w) = -R(y, x, z, w)$
(2) $R(x, y, z, w) = R(z, w, y, x)$
(3) $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$

for all $x, y, z, w \in V$.

George Washington University
California State University, San Bernardino
E-mail address: tzamorz@gwu.edu.

Furthermore, the set of algebraic curvature tensors, denoted $\mathcal{A}(V)$ is itself a vector space.

We now choose to solely focus on tensors themselves and derive results about them which could then be interpreted geometrically by others.

We now introduce the notion of a kernel of an algebraic curvature tensor.

Definition 1.2. The *kernel* of an algebraic curvature tensor, denoted $\operatorname{Ker}(R)$ is

$$\operatorname{Ker}(R) = \{x \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V\}.$$

However, Dunn, Franks, and Palmer [3] showed that

$$\begin{aligned}\operatorname{Ker}(R) &= \{x \in V : R(x, y, z, w) = 0 \text{ for all } y, z, w \in V\} \\ &= \{y \in V : R(x, y, z, w) = 0 \text{ for all } x, z, w \in V\} \\ &= \{z \in V : R(x, y, z, w) = 0 \text{ for all } x, y, w \in V\} \\ &= \{w \in V : R(x, y, z, w) = 0 \text{ for all } x, y, z \in V\},\end{aligned}$$

so for convention, we will only refer to the kernal as the set of vectors in the first slot that meet this annihilation condition.

1.2. **Symmetric and Antisymmetric Bilinear Forms.**

We will now introduce the idea of symmetrically and antisymmetrically built curvature tensors.

Definition 1.3. The mapping $\varphi : V \times V \to \mathbb{R}$ is a *symmetric bilinear form* if

(1) $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$, and
(2) $\varphi(x, y) = \varphi(y, x)$.

A bilinear form τ is *antisymmetric* if $\tau(x, y) = -\tau(y, x)$.

The set of symmetric bilinear forms on V is denoted $S^2(V^*)$ while the set of antisymmetric bilinear forms on V is denoted $\Lambda^2(V^*)$. These mappings give us a way to construct algebraic curvature tensors.

Definition 1.4. For $\varphi \in S^2(V^*)$ and $\tau \in \Lambda^2(V^*)$, R is a *canonical algebraic curvature tensor*, denoted R_φ or R_τ respectively if

(1) $R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w)$, or
(2) $R_\tau(x, y, z, w) = \tau(x, w)\tau(y, z) - \tau(x, z)\tau(y, w) - 2\tau(x, y)\tau(z, w)$.

It is known [4] that $R \in \mathcal{A}(V)$ can be expressed as sums of canonical algebraic curvature tensors. For the remainder of this research, our study will focus on sums of canonically built algebraic curvature tensors. From these definitions, we can construct algebraic curvature tensors using linear transformations.

Definition 1.5. Let $A, B : V \to V$ be linear transformations and let φ be an inner product on V. Then we define

(1) $R_A(x, y, z, w) = \varphi(Ax, w)\varphi(Ay, z) - \varphi(Ax, z)\varphi(Ay, w)$, and
(2) $R_B(x, y, z, w) = \varphi(Bx, w)\varphi(By, z) - \varphi(Bx, z)\varphi(By, w) - 2\varphi(Bx, y)\varphi(Bz, w)$

for all $x, y, z, w \in V$.

It is important to note that R_A and R_B are not necessarily algebraic curvature tensors for any A and B. For this to be the case, the concept of the adjoint must be introduced. The adjoint of A, denoted A^* with respect to the inner product φ is characterized by the equation $\varphi(Ax, y) = \varphi(x, A^*y)$. We say A is *symmetric* or *self-adjoint* if $A = A^*$ and that A is *skew-symmetric* or *skew-adjoint* if $A = -A^*$.

Contextualizing our definition, it is known in most cases by [2] that R_A is an algebraic curvature tensor if and only if A is self-adjoint with respect to the inner product on V and R_B is an algebraic curvature tensor if and only if B is skew-adjoint with respect to the inner product.

Another result that is important to the content of this work is that if $\operatorname{rank}(A) \geq 2$, then $\operatorname{Ker}(A) = \operatorname{Ker}(R_A)$, and if A has rank 1 or less, then $R_A = 0$, so $\operatorname{Ker}(A) \subseteq \operatorname{Ker}(R_A)$ [5]. Again, this is only when A is symmetric or skew-symmetric with respect to the inner product. For each linear transformation, we can construct an inner product such that A is self-adjoint which will be developed further in the next section.

1.3. **The Jordan Canonical Form of Linear Transformations and the Construction of the Inner Product.**

Work in [6,8,9] look at dependence of symmetrically and antisymetrically built algebraic curvature tensors. However, the research presented here ignores the condition that linear transformations are diagonalizable or block diagonalizable. Instead, we make a generalization to certain Jordan canonical forms of linear transformations, and introduce the potential area of study to include generalizations to any Jordan types, of which all linear transformations have a unique representation.

We introduce the notation

$$A = J(\lambda_1, t_1) \oplus J(\lambda_2, t_2) \oplus \cdots \oplus J(\lambda_k, t_k) \oplus C = \bigoplus_{i=1}^{k} J(\lambda_i, t_i) \oplus C$$

to represent a compact way of expressing linear transformations over real vector spaces in their Jordan canonical form, where $J(\lambda_i, t_i)$ represents a Jordan block of size t_i with real eigenvalue λ_i. The variable C denotes complex Jordan blocks. This study focuses only on Jordan forms that have real Jordan block. This notation still ensures that Jordan types are unique up to permutation of the Jordan bloacks. We also assume that each $\lambda_i \in \mathbb{R}$, but further results may be found when considering linear transformations over a complex vector space.

With this information, we can now construct an inner product from this transformation.

Definition 1.6. An *inner product* φ on a real vector space V is a nondegenerate symmetric bilinear form. That is, if

$$\varphi(v, w) = 0 \text{ for all } w \in V, \text{ then } v = 0$$

An inner product on V is called a *pseudo-euclidean* if it is not positive definite. Consequently, V is a *pseudo-eucliean vector space.*

Work by [6,8,9] only discusses linear dependence of symmetric and asymmetric algebraic curvature tensors with diagonalizable linear operators and a *positive definite* inner product. That is, $\varphi(v, v) \geq 0$ and $\varphi(v, v) = 0$ if and only if $v = 0$ for all $v \in V$. This research generalizes previous work by studying linear transformations with any Jordan type over pseudo-euclidean vector spaces.

Mal'cev [7] as well as Ahdout and Rothman [1] showed that every nondiagonalizable linear transformation is self-adjoint with respect to a unique pseudo-euclidean inner product up to signs. For $\dim(V) = n$ and

$$A = J(\lambda_1, t_1) \oplus J(\lambda_2, t_2) \oplus \cdots \oplus J(\lambda_k, t_k),$$

We write the inner product as

$$[\varphi_A] = \begin{bmatrix} \begin{matrix} 0 & \cdots & \varepsilon_1 \\ \vdots & \ddots & \vdots \\ \varepsilon_1 & \cdots & 0 \end{matrix} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \begin{matrix} 0 & \cdots & \varepsilon_k \\ \vdots & \ddots & \vdots \\ \varepsilon_k & \cdots & 0 \end{matrix} \end{bmatrix}$$

Where each block of ε_i corresponds to the standard involutary permutation matrix of size t_i. If n is even then each $\varepsilon_i = 1$. If n is odd, then $\varepsilon_i = \pm 1$. We will use this construction of the inner product for results on linear independence.

1.4. **Directed Graphs and Chain Complexes over Vector Spaces.**

Much of this research is concerned with chain complexes, which can be represented by directed graphs.

Definition 1.7. A *directed graph* is a pair $G = (V, E)$ where V is a set whose elements are called *vertices*, and E is a set of **ordered** pairs of vertices.

It is important that edges are ordered so we can assign orientation to edges as going from one vertex to another. We can now easily define a chain complex using the language of directed graph.

Definition 1.8. A *chain complex* over a vector space V with linear transformations $A, B : V \to V$ is a edge labelled directed graph, where each vertex is an instance of the vector space V, and that for each instance of the following subgraph

$$\bullet \cdots\cdots \bullet \xrightarrow{\quad A \quad} \bullet \xrightarrow{\quad B \quad} \bullet \cdots\cdots \bullet$$

we have $\mathrm{Im}\,(A) \subseteq \mathrm{Ker}\,(B)$.

Firstly, we observe that if $\mathrm{Im}\,(A) \subseteq \mathrm{Ker}\,(B)$, then $BA = 0$ which will play importance for many of the results in this study.

2. Context and Motivation

2.1. **Literature.**

McMahon [8] studied independence relationships of combinations of symmetric and antisymmetric algebraic curvature tensors. She was the first to introduce the idea of chain complexes to attempt to understand dependence relationships between precomposed symmetric and antisymmetric tensors. It was a chain complex in this study that inspired many of the results on independence and decomposability in this research.

Williams [9] continued to develop the language of chain complexes and directed graphs and began to question the role of graph-theoretic properties when discussing chain complexes associated with sums of canonical algebraic curvature tensors.

Julie [6] expands David's work by introducing the notion of a *weighted directed graph*, in that each vertex in the chain complex is assigned a weight that expresses an upper bound for the rank of the composition of two linear operators on each incident edge. She then attempted to find dependence relationships between tensors extracted from the directed graphs.

Each of these studies focuses mainly on dependence by *precomposition* by a linear operator. In other words, if A is a linear transformation and B is a self-or-skew-adjoint linear operator, then precomposing by A, denoted $A^{\dagger}$, would be the operation

$$A^{\dagger}R_B(x,y,z,w) = R_B(Ax,Ay,Az,Aw).$$

This line of inquiry is not further developed in this study. Rather, we take a different approach to studying directed graphs as chain complexes.

Additionally, [6, 8, 9] look mainly at the assumption that some operators are invertible. This research generalizes these concepts to consider non-diagonalizable linear transformations, with different Jordan canonical forms. Moreover, the focus on diagonalizability forgoes the posibillity that the inner product is not positive definite. This study uses non-positive definite inner products to ensure that linear transformations remain either self-adjoint or skew adjoint. Otherwise, the algebraic curvature tensors constructed from those operators would fail to satisfy the properties that define them. Furthermore, this research only focuses on the condition that $\mathrm{Im}\,(A) \subseteq \mathrm{Ker}\,(B)$ unlike [6] who considers compositions of related linear transformations of rank larger than zero.

Finally, this study uses the combinatorial approach developed by previous work to gain insight not only to linear dependence of tensors, but also decomposability of tensors. While similar graph-theoretic language is used, the implications vary on what results we wish to obtain, and the results on independence decomposability have different geometric implications.

2.2. **Relationship Between Images and Kernels.**

Given the condition of linear operators associated with chain complexes, we state a lemma that will prove useful in later results.

Lemma 2.1. *Let $A, B : V \to V$ be symmetric or skew-symmetric linear transformations. If $Im\,(A) \subseteq Ker\,(B)$, then $Im\,(B) \subseteq Ker\,(A)$.*

Proof. Given that $\mathrm{Im}\,(A) \subseteq \mathrm{Ker}\,(B)$, we know that $BA = 0$. Then, $(BA)^* = 0$, so

$$\begin{aligned} A^*B^* &= 0 \\ (\pm A)(\pm B) &= 0 \\ \pm AB &= 0 \\ AB &= 0, \end{aligned}$$

so $\mathrm{Im}\,(B) \subseteq \mathrm{Ker}\,(A)$, as desired. □

The chain complexes we study are structured so that each edge has the same orientation. This lemma is helpful in that it will aloow for the consideration of chain complexes that are obtained by reversing the orientation of each edge in the graph.

2.3. **A Combinatorial approach to Algebraic Curvature Tensors.**
The motivation to apply combinatorial methods to algebraic results is that of efficiency. This study acts as a stepping stone to build combinatorial intuition that can be applied to finite sets of algebraic curvature tensors.

Using directed graphs allows one to easily deduce information about the tensors associated with a given chain complex, and using graph theoretic ideas can shed further insight on the nature of how these tensors behave. Furthermore, one could potentially use graph theory and combinatorics to characterize particular sets of curvature tensors in relation to purely graph-theoretic properties, and potentially find new graph or tensor invariants.

2.4. **Independence and Dependence.**
Firstly, we know that the set of algebraic curvature tensors for a vector space, so we can treat these tensors as vectors in an attempt to study their linear independence or linear dependence. Furthermore, linear dependence of a sum of canonical algebraic curvature tensors such as

$$c_1R_1 + c_2R_2 + \cdots + c_kR_k = 0$$

reduces to the question

$$R_1 + \varepsilon_2R_2 + \cdots + \varepsilon_kR_k = 0,$$

where $\varepsilon_i = \pm 1$. This is becasue of the following observational result:

Proposition 2.2. *For $R_\varphi \in \mathcal{A}(V)$ and $c \in \mathbb{R}$, $R_\varphi = R_{-\varphi}$, and $cR_\varphi(x,y,z,w) = \operatorname{sign}(c)R_{\sqrt{|c|}\varphi}(x,y,z,w)$ for all $x,y,z,w \in V$.*

2.5. **Decomposability.**
We define decomposability on the direct sum of a vector space $V = V_1 \oplus V_2$.

Definition 2.3. An algebraic curvature tensor R is *decomposable*, denoted $R = S_1 \oplus S_2$ onto $V = V_1 \oplus V_2$ with $S_1 \in \mathcal{A}(V_1)$ and $S_2 \in \mathcal{A}(V_2)$ if

$$R(x_1+x_2, y_1+y_2, z_1+z_2, w_1+w_2) = S_1(x_1,y_1,z_1,w_1) + S_2(x_2,y_2,z_2,w_2).$$

This is true if and only if $v_1 \in V_1$ and $v_2 \in V_2$ imply $R(v_1, v_2, a, b) = 0$ for $a, b \in V$.

We will later extend this definition to include a notion of partial decomposability.

3. Augmentation of Sets of Linearly Independent Curvature Tensors

This section focuses of results of linear independence of curvature tensor. The results use the hypothesis that $A = J(0,t) \oplus 0$, that is, a single Jordan block of size t. Many other Jordan types can be hypothesized and this may lead to drastically different results. In the mean time, we prove the following results.

Theorem 3.1. *Suppose $A, B_1, \ldots, B_k : V \to V$ are linear transformations over a finite dimensional real vector space, such that $A = J(0,t) \oplus 0$, $t \geq 3$, $A = A^*$ and $B_i = \pm B_i^*$ with respect t the inner product φ on A. Let*

$$\begin{tikzcd} & & \bullet \\ \bullet \arrow[r, "A"] & \bullet \arrow[ur, "B_1"] \arrow[dr, "B_k"'] & \vdots \\ & & \bullet \end{tikzcd}$$

be a chain complex on V. If $\{R_{B_1}, \ldots, R_{B_k}\}$ is linearly independent, then $\{R_{B_1}, \ldots, R_{B_k}\} \cup \{R_A\}$ is linearly independent.

Proof. Let $\{e_1, \ldots, e_n\}$ be a Jordan Basis for A. Firstly, we know that $\operatorname{rank}(A) = t-1 \geq 2$, and from the chain complex, $\operatorname{Im}(A) \subseteq \operatorname{Ker}(B_i)$. Therefore $\dim(\operatorname{Ker}(B_i)) \geq \operatorname{rank}(A)$, so $\dim(\operatorname{Ker}(B_i)) \geq t-1$. On this basis, $\operatorname{span}\{e_1, \ldots, e_{t-1}\} \subseteq \operatorname{Ker}(B_i)$.

There are two cases that must be accounted for. Firstly, suppose $\{R_{B_1}, \ldots, R_{B_k}\}$ is independent and $cR_A + \sum_{i=1}^{k} c_i R_{B_i} = 0$.

The first case is when $c = 0$. Since, $\{R_{B_1}, \ldots, R_{B_k}\}$ is independent, each $c_i = 0$ as well so there is no nontrivial solution to $cR_A + \sum_{i=1}^{k} c_i R_{B_i} = 0$. Thus, the desired set is independent.

Now suppose $c \neq 0$ and there is some nontrivial solution to the equation

$$cR_A + \sum_{i=1}^{k} c_i R_{B_i} = 0.$$

We can divide by c to get

$$R_A + \sum_{i=1}^{k} \tilde{c}_i R_{B_i} = 0$$

where $\tilde{c}_i = \dfrac{c_i}{c}$. Now we can define $\tilde{B}_i = \sqrt{|\tilde{c}_i|}B$, so

$$R_A + \sum_{i=1}^{k} \varepsilon_i R_{\tilde{B}_i} = 0$$

by Proposition 2.2. Also, $c_i = 0$ if and only if $\tilde{c}_i = 0$, so we can reduce the equation even further to

$$R_A + \sum_{c_i \neq 0} \varepsilon_i R_{\tilde{B}_i} = 0.$$

Notice that at least one of the $\tilde{c}_i \neq 0$, since otherwise $R_A = 0$, which is impossible as $\operatorname{rank}(A) \geq 2$. In this way, the sum of the $R_{\tilde{B}_i}$ is nonempty.

Claim. *There exist basis vectors e_p, e_q, e_r, e_s such that $R_A(e_p, e_q, e_r, e_s) \neq 0$, and $R_{\tilde{B}_i}(e_p, e_q, e_r, e_s) = 0$ for $1 \leq i \leq k$. This contradicts the existence of at least one nonzero c_i.*

Since $A = J(0,t) \oplus 0$, this gives the matrix

$$A = [0|e_1|e_2|\cdots|e_{t-1}|0|\cdots|0]\,,$$

so $Ae_j = Ae_{j-1}$ for $2 \leq j \leq t$ and $Ae_\ell = 0$ otherwise. The inner product with respect to A is

$$[\varphi_A] = \begin{bmatrix} 0 & \cdots & \varepsilon_t & & & \\ \vdots & \ddots & \vdots & & 0 & \\ \varepsilon_t & \cdots & 0 & & & \\ & & & \varepsilon_{t+1} & & \\ & 0 & & & \ddots & \\ & & & & & \varepsilon_n \end{bmatrix}.$$

Firstly $\varphi(e_1, e_2) = 0$ since $t \geq 3$, so $\varphi(e_c, e_{t-c+1}) = \varepsilon_t$ where $\varepsilon_t = \pm 1$ for $1 \leq c \leq t$. Moreover, $\varphi(e_d, e_d) = \varepsilon_d$ for $t+1 \leq d \leq n$.

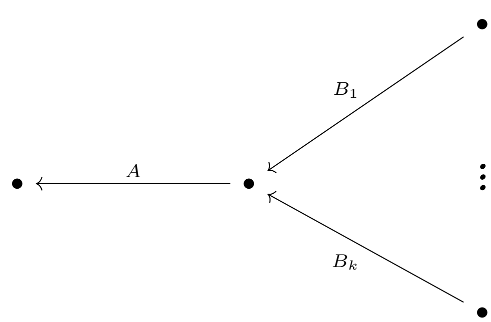
We see that

$$R_A(e_2, e_t, e_2, e_t) = \varphi(Ae_2, e_t)\varphi(Ae_t, e_2) - \varphi(Ae_2, e_2)\varphi(Ae_t, e_t) = 1.$$

However, $e_2 \in \operatorname{span}\{e_1, \ldots, e_t\} \subseteq \operatorname{Ker}(B_i)$, and since $\operatorname{Ker}\left(\tilde{B}_i\right) = \operatorname{Ker}(B_i) = \operatorname{Ker}(R_{B_i})$, this tells us $R_{\tilde{B}_i}(e_2, e_t, e_2, e_t) = 0$ for $i = 1, \ldots, k$. Therefore, there is no dependence relationship in the set $\{R_A, R_{B_1}, \ldots, R_{B_k}\}$ if $\{R_{B_1}, \ldots, R_{B_k}\}$ is independent. □

By Lemma 2.1, there is an immediate corollary.

Corollary 3.2. *Suppose $A, B_1, \ldots, B_k : V \to V$ are linear transformations over a finite dimensional real vector space, such that $A = J(0,t) \oplus 0$, $t \geq 3$, $A = A^*$ and $B_i = \pm B_i^*$ with respect t the inner product φ on A. Let*



be a chain complex on V. If $\{R_{B_1}, \ldots, R_{B_k}\}$ is linearly independent, then $\{R_{B_1}, \ldots, R_{B_k}\} \cup \{R_A\}$ is linearly independent.

These results can be expanded upon by changing the structure of the directed graphs, or by changing the Jordan type of the linear operators.

4. Decomposability of Curvature Tensors

We now give a characterization of decomposability of algebraic curvature tensors. This will aid in the formulation of results on decomposability of symmetric and anti-symmetric tensors that are expressed through chain complexes of linear operators.

Theorem 4.1. *Let $V = V_1 \oplus V_2$. An Algebraic Curvature Tensor R is decomposable into $R = S_1 \oplus S_2$ if and only if there are $R_1, R_2 \in \mathcal{A}(V)$ such that $R = R_1 + R_2$ and $V_1 \subseteq Ker(R_2)$ and $V_2 \subseteq Ker(R_1)$ where neither V_1 nor V_2 are zero-dimensional.*

Proof. Suppose $R = S_1 \oplus S_2$ where $S_1 \in \mathcal{A}(V_1)$ and $S_2 \in \mathcal{A}(V_2)$. We define

$$R_i(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) := S_i(x_i, y_i, z_i, w_i)$$

for $i = 1, 2$ and $R_i \in \mathcal{A}(V)$. Since $V = V_1 \oplus V_2$ each $v \in V$ can be written as $v = v_1 + v_2$ where $v_1 \in V_1$ and $v_2 \in V_2$. We see that

$$\begin{aligned} R(x, y, z, w) &= S_1(x_1, y_1, z_1, w_1) + S_2(x_2, y_2, z_2, w_2) \\ &= R_1(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) + R_2(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) \\ &= R_1(x, y, z, w) + R_2(x, y, z, w), \end{aligned}$$

so $R = R_1 + R_2$ where $R_1, R_2 \in \mathcal{A}(V)$. Additionally, for some $v_2 \in V_2$, we can express this vector as $v_2 = 0 + v_2$ with $0 \in V_1$. Consequently, for any $v_2 \in V_2$, $v_2 \in \operatorname{Ker}(R_1)$ since

$$R_1(v_2, p, q, r) = S_1(0, p_1, q_1, r_1) = 0$$

for any $p, q, r \in V$. Similarly, for any $v_1 \in V_1$, $v_1 \in \operatorname{Ker}(R_2)$. Therefore, $V_1 \subseteq \operatorname{Ker}(R_2)$ and $V_2 \subseteq \operatorname{Ker}(R_1)$.

Now suppose $R = R_1 + R_2$ with $V_1 \subseteq \operatorname{Ker}(R_2)$ and $V_2 \subseteq \operatorname{Ker}(R_1)$ for $R_1, R_2 \in \mathcal{A}(V)$. Define

$$S_i := R_i\Big|_{V_i} \in \mathcal{A}(V_i).$$

Claim. *$R = S_1 \oplus S_2$.*

To see this, note that

$$R(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) = \\ R_1(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) + R_2(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2),$$

and since $V_1 \subseteq \operatorname{Ker}(R_2)$ and $V_2 \subseteq \operatorname{Ker}(R_1)$,

$$\begin{aligned} &R_1(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) + R_2(x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2) \\ &= R_1(x_1, y_1, z_1, w_1) + R_2(x_2, y_2, z_2, w_2) \\ &= R_1\Big|_{V_1}(x_1, y_1, z_1, w_1) + R_2\Big|_{V_2}(x_2, y_2, z_2, w_2) \\ &= S_1(x_1, y_1, z_1, w_1) + S_2(x_2, y_2, z_2, w_2). \end{aligned}$$

Therefore, $R = S_1 \oplus S_2$, as needed. □

With this comes an immediate corollary.

Corollary 4.2. *Let $V = V_1 \oplus \cdots \oplus V_k$ with $k \leq \left\lfloor \dfrac{\dim(V)}{2} \right\rfloor$. An Algebraic Curvature Tensor R is decomposable into $R = S_1 \oplus \cdots \oplus S_k$ if and only if there exist $R_1, \ldots, R_k \in \mathcal{A}(V)$ such that $R = \displaystyle\sum_{i=1}^{k} R_i$ and $V_i \subseteq \displaystyle\sum_{i \neq j} (Ker(R_j))$.*

We now contextualize our result on decomposability to a chain complex of linear operators over a finite dimensional real vector space

Theorem 4.3. *Let*

$$\begin{tikzcd}
 & & \bullet \\
\bullet \arrow[r, "A"] & \bullet \arrow[ur, "B_1"] \arrow[dr, "B_k"'] & \vdots \\
 & & \bullet
\end{tikzcd}$$

be a chain complex where A is diagonalizable, rank $(A) \geq 2$, and $B_i = \pm B_i^*$. Then $R = R_A + \sum_{i=1}^{k} R_{B_i}$ is decomposable into $R = R_A \oplus \sum_{i=1}^{k} R_{B_i}$.

Proof. Suppose rank $(A) = t$. Since A is diagonalizable, we can express it as

$$[A] = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & \text{\Large 0} & \\ & & \lambda_t & & & \\ & & & 0 & & \\ & \text{\Large 0} & & & \ddots & \\ & & & & & 0 \end{bmatrix}.$$

This gives us a vector spaces decomposition $V = \mathrm{Im}\,(A) \oplus \mathrm{Ker}\,(A) = V_1 \oplus V_2$. By hypothesis, $\mathrm{Im}\,(A) = V_1 \subseteq \mathrm{Ker}\,(B_i) = \mathrm{Ker}\,(R_{B_i})$. Let $R_B = \sum_{i=1}^{k} R_{B_i}$. It is clear that $V_1 \subseteq \mathrm{Ker}\,(R_B)$ because any $v_1 \in V_1$ annihilates each R_{B_i}. Also, $V_2 = \mathrm{Ker}\,(A) = \mathrm{Ker}\,(R_A)$, so R is decomposable into $R = R_A \oplus R_B = R_A \oplus \sum_{i=1}^{k} R_{B_i}$, as needed. $\square$

Again, by Lemma 2.1, we derive an immediate corollary.

Corollary 4.4. *Let*

$$\begin{tikzcd}
 & & \bullet \arrow[dl, "B_1"'] \\
\bullet & \bullet \arrow[l, "A"'] & \vdots \\
 & & \bullet \arrow[ul, "B_k"]
\end{tikzcd}$$

be a chain complex where A is diagonalizable, rank$(A) \geq 2$, and $B_i = \pm B_i^*$. Then $R = R_A + \sum_{i=1}^{k} R_{B_i}$ *is decomposable into* $R = R_A \oplus \sum_{i=1}^{k} R_{B_i}$.

5. k-Decomposability of Curvature Tensors

Decomposability of algebraic curvature tensors is a worthwhile endeavour, as it is a tool that allows for the breaking up of a tensor into smaller, more managable, pieces. Naturally, the notion of decomposability conveys an idea of a complete breakdown of a tensor. The question we ask is akin to asking if there is an extension of the concept of decomposability for tensors that do not break down completley.

This question formulates the idea of k-decomposability: a partial breakdown of a curvature tensor expressed as a sum of algebraic curvature tensors. This contextualizes decomposability as a special case of k-decomposability, namely 0-decomposability. Conceptually, a breakdown of a tensor is partial if tensors in the sum have non-trivial intersecting kernels. Crucially, decomposability of curvature tensors imples that the tensor kernels sum to the entire vector space, something that can now be manipulated for further study.

[6] conjectured about a possible definition of k-decomposability to allow for the condition that we overcount $\dim(V)$ through intersecting tensor kernels. We know give a formal, completed definition of k-decomposability.

Definition 5.1. An algebraic curvature tensor is *k-decomposable* over a finite dimensional vector space of dimension n, denoted $R = R_1 \oplus_k R_2$ if

(1) $R = R_1 + R_2$, and
(2) $\dim(\mathrm{Ker}\,(R_1)) + \dim(\mathrm{Ker}\,(R_2)) - \dim(\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2)) = n - k$.

However, this definition must be justified to be valid. The following result does this.

Theorem 5.2. *An algebraic curvature tensor is decomposable as $R = R_1 \oplus R_2$ if and only if $R = R_1 \oplus_0 R_2$.*

Proof. Suppose $R = R_1 \oplus R_2$. We know that $V = V_1 \oplus V_2$, and that $V_2 \subseteq \mathrm{Ker}\,(R_1)$ and $V_1 \subseteq \mathrm{Ker}\,(R_2)$. Also, $R = R_1 + R_2$, and since $V = V_1 + V_2$, it follows that $V = \mathrm{Ker}\,(R_1) + \mathrm{Ker}\,(R_2)$. Therefore,

$$\begin{aligned} \dim(\mathrm{Ker}\,(R_1) + \mathrm{Ker}\,(R_2)) &= \dim(V) \\ \dim(\mathrm{Ker}\,(R_1)) + \dim(\mathrm{Ker}\,(R_2)) - \dim(\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2)) &= n, \end{aligned}$$

so R is 0-decomposable.

Now suppose that R is 0-decomposable. Again, it follows that

$$\dim(\mathrm{Ker}\,(R_1)) + \dim(\mathrm{Ker}\,(R_2)) - \dim(\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2)) = \dim(\mathrm{Ker}\,(R_1) + \mathrm{Ker}\,(R_2)) = n,$$

so $V = \mathrm{Ker}\,(R_1) + \mathrm{Ker}\,(R_2)$. If $\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2) = \{0\}$, then the conclusion is satisifed, so assume there is a nontrivial kernel. Firstly, $\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2) \subseteq V$, as $\mathrm{Ker}\,(R_1) \subseteq V$ and $\mathrm{Ker}\,(R_2) \subseteq V$, so let $\{k_1, \ldots k_\ell\}$ be a basis for $\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2)$. Since $\mathrm{Ker}\,(R_1) \cap \mathrm{Ker}\,(R_2) \subseteq \mathrm{Ker}\,(R_2)$, we can extend this basis to $\{e_1, \ldots, e_p, k_1, \ldots k_\ell\}$, which is a basis for $\mathrm{Ker}\,(R_2)$. Let $V_1 = \mathrm{Ker}\,(R_2)$. We can extend this basis even further to $\{e_1, \ldots, e_p, k_1, \ldots k_\ell, f_1, \ldots, f_s\}$ which is a basis for V, so $\dim(V) = p + \ell + s = n$.

Now, each $f_i \in \{f_1, \dots, f_s\}$ can be written as $f_i = f_{i,1} + f_{i,2}$ with $f_{i,1} \in \operatorname{Ker}(R_1)$ and $f_{i,2} \in \operatorname{Ker}(R_2)$. This due to the fact that $V = \operatorname{Ker}(R_1) + \operatorname{Ker}(R_2)$. Let $f_{i,1} = \tilde{f}_i = f_i - f_{i,2} \in \operatorname{Ker}(R_1)$ and define $V_2 \coloneqq \operatorname{span}\left\{\tilde{f}_1, \dots, \tilde{f}_s\right\}$.

Claim. $V = V_1 \oplus V_2$

In order to prove this, it first needs to be shown that $\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\}$ is a basis for V. To do this, we need only show that $\operatorname{span}\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\} = V$.

Suppose $v \in V$. Since $\{e_1, \dots, e_p, k_1, \dots k_\ell, f_1, \dots, f_s\}$ is a basis for V, there exist coefficients such that

$$\begin{aligned} v &= \sum_{i=1}^{p} \alpha_i e_i + \sum_{j=1}^{\ell} \beta_j k_j + \sum_{m=1}^{s} \gamma_m f_m \\ &= \sum_{i=1}^{p} \alpha_i e_i + \sum_{j=1}^{\ell} \beta_j k_j + \sum_{m=1}^{s} \gamma_m \tilde{f_m} + \sum_{m=1}^{s} \gamma_m f_{m,2}. \end{aligned}$$

We know $f_{i,2} \in \operatorname{Ker}(R_2) = \operatorname{span}\{e_1, \dots, e_p, k_1, \dots k_\ell\}$, so

$$f_{i,2} = \sum_{r=1}^{p} \delta_r e_r + \sum_{s=1}^{l} \varepsilon_s k_s.$$

Therefore,

$$v = \sum_{i=1}^{p} \alpha_i e_i + \sum_{j=1}^{\ell} \beta_j k_j + \sum_{m=1}^{s} \gamma_m (\sum_{r=1}^{p} \delta_r e_r + \sum_{s=1}^{\ell} \varepsilon_s k_s) + \sum_{m=1}^{s} \gamma_m \tilde{f_m},$$

where $\sum_{i=1}^{p} \alpha_i e_i + \sum_{j=1}^{\ell} \beta_j k_j + \sum_{m=1}^{s} \gamma_m (\sum_{r=1}^{p} \delta_r e_r + \sum_{s=1}^{\ell} \varepsilon_s k_s) \in \operatorname{span}\{e_1, \dots, e_p, k_1, \dots k_\ell\}$, and $\sum_{m=1}^{s} \gamma_m \tilde{f_m} \in \operatorname{span}\left\{\tilde{f}_1, \dots, \tilde{f}_s\right\}$. So $v \in \operatorname{span}\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\}$. Since $\left|\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\}\right| = p+l+s = n$, and $\operatorname{span}\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\} = V$, we conclude that $\left\{e_1, \dots, e_p, k_1, \dots k_\ell, \tilde{f}_1, \dots, \tilde{f}_s\right\}$ is a basis for V. Now, by construction $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$, so $V = V_1 \oplus V_2$.

Moreover, since $V = V_1 \oplus V_2$ and $V_1 \subseteq \operatorname{Ker}(R_2)$ and $V_2 \subseteq \operatorname{Ker}(R_1)$, $R = R_1 \oplus R_2$, as desired. $\square$

We refer the readers to Conjectures 6.3 and 6.4 to see potential contextualizations of this result.

6. Conjectures and Further Research

There are many potentiaal lines of inquiry that can be followed from the implications of this research. We focus on combinatorial and algebriac extensions rather than exploring the geometric implications.

Firstly, the chain complex of desire in this study can be a connected subgraph of a more expansive graph of the same type:

Definition 6.1. An *arborescence* is a directed graph in which, for a single vertex u called the *root* and any other vertex v there is exactly one directed path from u to v.

We use the graph-theoretic concept of an *proper edge-coloring*, which is a mapping $c : E \to \mathbb{N}$ that assigns each edge $e \in E$ to a natural number $c(e)$ refered to as a *color*,such that if e and f are incident with the same vertex, then $c(e) \neq c(f)$ A k-*coloring* is a proper coloring that uses k colors. We formulate the following conjecture.

Conjecture 6.2. *Let G an arborescence with a 2-edge coloring such that each path in G is alternating. If G is a chain complex such that edges corresponding to linear operators $A_1, \dots, A_s$ have color c_1 and edges corresponding to linear operators $B_1, \dots, B_t$ have color c_2, then*

$$R = \sum_{i=1}^{s} R_{A_i} + \sum_{j=1}^{t} R_{B_j}$$

is decomposable into

$$R = \sum_{i=1}^{s} R_{A_i} \oplus \sum_{j=1}^{t} R_{B_j}.$$

We now state potential results that ponder the significance of the Jordan type of A with respect to the chain complex of study.

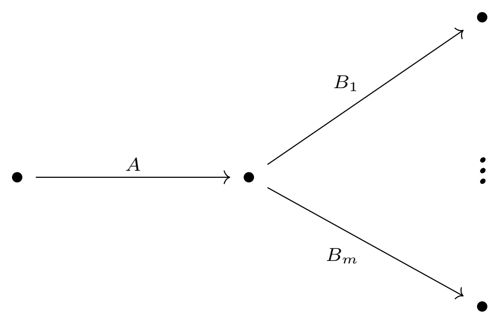
Conjecture 6.3. *Suppose $A, B_1, \dots, B_k : V \to V$ are linear transformations over a finite dimensional real vector space, such that $A = J(0,t) \oplus 0$, $t \geq 3$, $A = A^*$ and $B_i = \pm B_i^*$ with respect t the inner product φ on A. Let*

$$\begin{tikzcd} & & \bullet \\ \bullet \arrow[r, "A"] & \bullet \arrow[ur, "B_1"] \arrow[dr, "B_k"'] & \vdots \\ & & \bullet \end{tikzcd}$$

be a chain complex on V. Then $R = R_A \oplus_1 \sum_{i=1}^{k} R_{B_i}$.

We now extend this idea to a potentially more general reseult:

Conjecture 6.4. *Let*

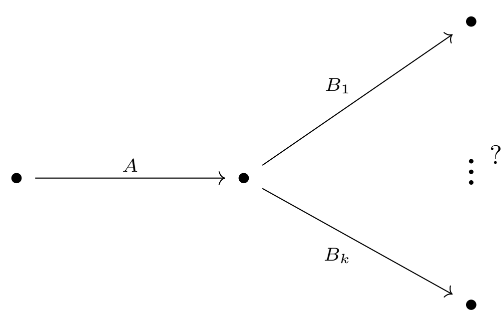


be a chain complex on V, and let $A = A^*$, $B = \pm B^*$ with respect to the inner product φ. Suppose $A = \bigoplus_{i=1}^{s} J(\lambda_i, t_i)$ where all Jordan blocks are real such that there are k Jordan blocks $J(\lambda_i, t_i)$ with $\lambda_i = 0$ and $t_i \geq 2$. Then

$$R = R_A \oplus_k \sum_{i=1}^{m} R_{B_i}.$$

We now ask general questions that can drive further research in this topic

(1) Can results on linear independence, decomposability, and k-decomposability be generalized to include linear operators with complex Jordan types? Does this change the results found for the chain complex



(2) Can graph-theoretic properties illicit similar results on independence and decomposability of algebraic curvature tensors? Can results be generalized to all types of acyclic directed graphs? Can results be found for graph containing at least one directed cycle?
(3) Given that chain complexes can be used to develop results on independence and decomposability, does this suggest a relationship between linear independence and decomposability of canonical algebraic curvature tensors?
(4) Can we extend our notion of decomposability to consider tensors over the quotient space $\overline{V} = V/K$ where

$$K = \bigcap_{i=1}^{k} \mathrm{Ker}\,(R_i)\,.$$

where $R = \sum_{i=1}^{k} R_i$.

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