A Cavalcade of Careless Computation

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Abstract

In this paper, we investigate properties of canonical algebraic curvature tensors arising from the Ricci-Weyl decomposition. We define the Ricci map on symmetric bilinear forms and use it to provide a simple construction for all canonical Einstein tensors. Finally, we characterize when canonical tensors are Weyl-flat.

1 Introduction

Let (V, g) be a real vector space V of dimension $n > 2$ equipped with a non-degenerate symmetric bilinear form g. Let $S^2(V^*)$ denote the space of symmetric bilinear forms over V.

Definition 1. An *algebraic curvature tensor* is a tensor $R \in \otimes^4 V^*$ that satisfies the following identities:

(a) $R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y)$.

(b) $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

The vector space of algebraic curvature tensors over V is denoted by $\mathcal{A}(V)$.

Every algebraic curvature tensor can be realized as the curvature tensor at a point of a pseudo-Riemannian manifold ([1], Lem. 6.1.1).

Definition 2. Let $\phi, \psi \in S^2(V^*)$. The *Kulkarni-Nomizu product* $\phi \owedge \psi$ is given by

$$\phi \owedge \psi(x, y, z, w) = \phi(x, z)\psi(y, w) + \phi(y, w)\psi(x, z) - \phi(x, w)\psi(y, z) - \phi(y, z)\psi(x, w).$$

Definition 3. Let $\phi \in S^2(V^*)$. We define an associated *canonical algebraic curvature tensor* $R_\phi = -\frac{1}{2}\phi \owedge \phi$.

Our primary interest in canonical tensors arises from the following result:

Proposition 1 ([1], Thm. 1.6.1)**.**

$$\mathcal{A}(V) = \operatorname{span}\{R_\phi : \phi \in S^2(V^*)\}.$$

We study canonical tensors using the decomposition of $\mathcal{A}(V)$ which is induced by metric contraction.

Definition 4. Let $\{e_i\}$ be a basis for V and define $G = [g(e_i, e_j)]$. The *Ricci tensor* of $R \in \mathcal{A}(V)$ is the metric contraction of R given by

$$\mathrm{Ric}(R)_{xy} = \sum_{i,j} G^{-1}_{ij} R(x, e_i, e_j, y).$$

Note that this contraction is basis-independent [1].

Definition 5. $R \in \mathcal{A}(V)$ is a *Weyl tensor* if $\mathrm{Ric}(R) = 0$. The space of Weyl tensors over V is denoted $\mathcal{W}(V)$.

The space of algebraic curvature tensors then decomposes via the following map:

Definition 6. We define the map $\sigma : S^2(V^*) \to \mathcal{A}(V)$ by

$$\sigma(\phi) = -\frac{1}{n-2} g \owedge \phi + \frac{\operatorname{Tr}\phi}{2(n-1)(n-2)} g \owedge g.$$

Proposition 2 ([1], Thm. 4.1.1)**.** *The map σ is an injective linear map decomposing the space of algebraic curvature tensors as $\mathcal{A}(V) = \mathcal{W}(V) \oplus \operatorname{Im}\sigma$.*

In Section 2, we will derive basic properties of the Ricci tensors of canonical algebraic curvature tensors. In Section 3, we then characterize all the canonical tensors which have Weyl component zero.

2 The Ricci Map

Proposition 3. *Let $\{e_i\}$ be a basis for V and let $\phi, \psi \in S^2(V^*)$. Let $\Phi = [\phi(e_i, e_j)], \Psi = [\psi(e_i, e_j)]$, and $G = [g(e_i, e_j)]$ be matrix representations of ϕ, ψ, and g respectively. Then*

$$[\mathrm{Ric}(\phi \owedge \psi)] = \Phi(G^{-1}\Psi - \operatorname{Tr}(G^{-1}\Psi)\,\mathrm{Id}) + \Psi(G^{-1}\Phi - \operatorname{Tr}(G^{-1}\Phi)\,\mathrm{Id}).$$

Proof. We compute

$$\begin{aligned}
\mathrm{Ric}(\phi \owedge \psi)_{il} &= \sum_{j,k} G^{-1}_{jk}(\Phi_{ik}\Psi_{jl} + \Phi_{jl}\Psi_{ik} - \Phi_{il}\Psi_{jk} - \Phi_{jk}\Psi_{il}) \\
&= \sum_{j,k} \Phi_{ik} G^{-1}_{kj} \Psi_{jl} + \sum_{j,k} \Psi_{ik} G^{-1}_{kj} \Phi_{jl} - \Phi_{il} \sum_{j,k} G^{-1}_{kj} \Psi_{jk} - \Psi_{il} \sum_{j,k} G^{-1}_{kj} \Phi_{jk} \\
&= (\Phi G^{-1} \Psi)_{il} + (\Psi G^{-1} \Phi)_{il} - \Phi_{il} \operatorname{Tr}(G^{-1}\Psi) - \Psi_{il} \operatorname{Tr}(G^{-1}\Phi),
\end{aligned}$$

from which the desired result is immediate. □

We will suppose that g is positive-definite for the remainder of the paper.

Corollary 3.1. *If $\{e_i\}$ is an orthonormal basis with respect to g, then (using the same notation for bilinear forms and their matrix representations) we have*

$$\begin{aligned}
\mathrm{Ric}(\phi \owedge \phi) &= 2\phi(\phi - (\operatorname{Tr}\phi)\,\mathrm{Id}) \\
\mathrm{Ric}(g \owedge \phi) &= (2-n)\phi - (\operatorname{Tr}\phi)\,\mathrm{Id} \\
\mathrm{Ric}(g \owedge g) &= 2(1-n)\,\mathrm{Id}
\end{aligned}$$

Definition 7. We define the *Ricci map* $\rho : S^2(V^*) \to S^2(V^*)$ by $\phi \mapsto \operatorname{Ric} R_\phi$, which by the above computation yields $\rho(\phi) = \phi((\operatorname{Tr}\phi)\operatorname{Id} - \phi)$ over an orthonormal basis.

We will now derive some properties of the Ricci map. Let $\phi \in S^2(V^*)$. By the Spectral Theorem, let $\{e_i\}$ be an orthonormal basis diagonalizing ϕ. Let $\lambda_i = \phi_{ii}$ be the eigenvalues of ϕ and let $\lambda = \sum_i \lambda_i = \operatorname{Tr}\phi$ be the trace. Then $\rho(\phi)_{ii} = \lambda_i(\lambda - \lambda_i)$.

Lemma 4. $\rho(\phi) = 0 \iff \operatorname{rk}\phi \leq 1$.

Proof. If $\rho(\phi) = 0$ but $\phi \neq 0$, then $\forall i$ with $\lambda_i \neq 0$, we have $\lambda = \lambda_i$. So there is only one nonzero eigenvalue α, which we suppose has multiplicity k. Then $\lambda = k\alpha \implies (k-1)\alpha = 0 \implies k = 1$. Thus, $\operatorname{rk}\phi = 1$. The converse is easy to check. □

Lemma 5. $\rho(\phi) = \phi \iff \phi = \frac{1}{k-1}\operatorname{Id}_k \oplus 0$ *for some* $2 \leq k \leq n$.

Proof. If $\lambda_i \neq 0$, then $\lambda - \lambda_i = 1$, so ϕ has at most one distinct nonzero eigenvalue α. Letting k be the multiplicity of α, we have $\alpha k - \alpha = 1$, giving the desired result. A straightfoward computation shows the converse. □

Definition 8. $R \in \mathcal{A}(V)$ is an *Einstein tensor* if there exists $c \in \mathbb{R}$ such that $\operatorname{Ric} R = cg$.

Shapiro [2] classified all forms $\phi \in S^2(V^*)$ for which R_ϕ is Einstein. We provide the same result with a streamlined proof using the Ricci map.

Proposition 6. *Let* $c \neq 0$. *If* $c > 0$, *then* $\rho(\phi) = cg \iff \phi = \pm\frac{c}{2(n-1)}g$. *If* $c < 0$, *then* $\rho(\phi) = cg \iff \phi = \pm\sqrt{-c}\left(\sqrt{\frac{l-1}{k-1}}\operatorname{Id}_k \oplus -\sqrt{\frac{k-1}{l-1}}\operatorname{Id}_l\right)$.

Proof. For all i, we have $\lambda_i(\lambda - \lambda_i) = c$. This is a quadratic in λ_i, so it has at most two solutions. That is, $\lambda_i \in \{\alpha, \beta\}$ with $\alpha + \beta = \lambda$ and $\alpha\beta = c$.

If ϕ has only one eigenvalue, then it is easily checked that $\phi = \pm\frac{c}{2(n-1)}g$ and that $c > 0$. If ϕ has two eigenvalues, then without loss of generality we write $\phi = \alpha\operatorname{Id}_k \oplus \beta\operatorname{Id}_l$. The relation $\alpha + \beta = \lambda$ necessitates that $\alpha(k-1) + \beta(l-1) = 0$. Note that this requires the eigenvalues to have different signs, so $c < 0$. Substituting $\beta = \frac{c}{\alpha}$, we obtain $\alpha = \pm\sqrt{-c\frac{l-1}{k-1}}$ and symmetrically $\beta = \mp\sqrt{-c\frac{k-1}{l-1}}$. □

The canonical Einstein tensors with Einstein constant zero are all trivial by Lemma 4.

3 Ricci-Weyl Decomposition of Canonical ACTs

We first show a general diagonalization result for canonical tensors as a technical aid.

Definition 9. $R \in \mathcal{A}(V)$ is called *pure* if there exists an orthonormal basis $\{e_i\}$ for V such that $R_{ijkl} = 0$ when $|\{i,j,k,l\}| > 2$. We then say that R is pure on $\{e_i\}$.

Lemma 7. *Suppose that* $\phi, \psi \in S^2(V^*)$ *are simultaneously diagonal on a basis* $\{e_i\}$. *Then* $(\phi \owedge \psi)_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})(\phi_{ii}\psi_{jj} + \phi_{jj}\psi_{ii})$, *where* δ_{ij} *is the Kronecker delta.*

Proof. Note that $\forall i, j$, we have $\phi_{ij} = \delta_{ij}\phi_{ii}$ and $\psi_{ij} = \delta_{ij}\psi_{ii}$. The result is then directly computed. $\square$

Lemma 8. *Let $\phi \in S^2(V^*)$ with $\operatorname{rk}\phi > 2$. Then R_ϕ is pure on $\{e_i\}$ if and only if ϕ is diagonal on $\{e_i\}$.*

Proof. If ϕ is diagonal, then the previous lemma shows that R_ϕ is pure.

Now, assume that R_ϕ is pure and suppose that all diagonal entries of ϕ are nonzero. Then either ϕ is diagonal or there exist $i \neq j$ such that $\phi_{ij} \neq 0$. Then $\forall k \notin \{i, j\}$,

$$0 = R_{ikkj} = \phi_{ij}\phi_{kk} - \phi_{ik}\phi_{kj} \implies \phi_{ik}, \phi_{kj} \neq 0,$$

so all entries of ϕ must be nonzero. Since $\operatorname{rk}\phi > 1$, there exist $i \neq j$ such that $R_{ijji} \neq 0$ and thus $\phi_{ii}\phi_{jj} \neq \phi_{ij}^2$. Then $\forall k \notin \{i, j\}$,

$$0 = R_{ijki} = \phi_{ii}\phi_{jk} - \phi_{ik}\phi_{ji} \implies \phi_{ik} = \phi_{jk}\frac{\phi_{ii}}{\phi_{ij}}$$
$$0 = R_{jikj} = \phi_{jj}\phi_{ik} - \phi_{jk}\phi_{ij} \implies \phi_{ik} = \phi_{jk}\frac{\phi_{ij}}{\phi_{jj}},$$

so $\phi_{ii}\phi_{jj} = \phi_{ij}^2$, a contradiction. Thus, ϕ is diagonal.

Now, suppose that a diagonal entry of ϕ is zero. Then without loss of generality, let $\phi_{11} = 0$. Then $\forall i \neq j$ with $1 \notin \{i, j\}$, we have

$$0 = R_{i11j} = \phi_{ij}\phi_{11} - \phi_{i1}\phi_{1j} = -\phi_{i1}\phi_{1j},$$

so either ϕ_{i1} or ϕ_{1j} must be zero. By symmetry of ϕ, there then exists at most one i such that $\phi_{i1} \neq 0$.

Suppose that there exists $i \neq 1$ such that $\phi_{i1} \neq 0$. Since $\operatorname{rk}\phi > 2$, there exist j, k such that $j \notin \{1, i\}$ and $\phi_{jk} \neq 0$, as otherwise $\phi_{i1} = \phi_{1i}$ and ϕ_{ii} would be the only nonzero entries of ϕ. Then

$$0 = R_{ij1k} = \phi_{ik}\phi_{j1} - \phi_{i1}\phi_{jk} = -\phi_{i1}\phi_{jk} \implies \phi_{jk} = 0,$$

a contradiction. Thus, $\phi = 0 \oplus \phi|_{\operatorname{span}\{e_2,\ldots,e_n\}}$. Repeating this process on all zero diagonal entries of ϕ, we find that $\phi = 0 \oplus \phi'$, where $\operatorname{rk}\phi' > 2$ and all diagonal entries of ϕ' are nonzero. As proven earlier, ϕ' must then be diagonal, so ϕ is likewise. $\square$

We now investigate the Weyl component of canonical tensors.

Lemma 9. *Every Weyl tensor can be written as a linear combination of pure Weyl tensors.*

Proof. If ϕ is diagonal on $\{e_i\}$, then so is $\rho(\phi)$ and thus $\sigma \circ \rho(\phi)$. But then W_ϕ is the difference of tensors that are pure on the same basis and itself pure.

Since every Weyl tensor can be written as a linear combination of canonical tensors, it can be written as a linear combination of the Weyl components of canonical tensors, which are pure. $\square$

Proposition 10. *If $n \geq 4$, then $R_\phi \in \operatorname{Im}\sigma$ if and only if there exists $\alpha \in \mathbb{R}$ and $\psi \in S^2(V^*)$ with $\operatorname{rk}\psi \leq 1$ such that $\phi = \alpha g + \psi$.*

Proof. If $\phi = ag + \psi$, then

$$\phi \owedge \phi = a^2 g \owedge g + 2g \owedge \psi \in \operatorname{Im}\sigma,$$

as $\psi \owedge \psi = 0$.

On the other hand, suppose that $\phi \notin \operatorname{span}\{g\}$ but $R_\phi \in \operatorname{Im}\sigma$. Then $R_\phi - \sigma \circ \rho(\phi) = 0$. Let $\{e_i\}$ be an orthonormal basis diagonalizing ϕ. Then $\forall i \neq j$, we have by Lemma 7 that

$$(n-2)\phi_{ii}\phi_{jj} - \phi_{ii}(\operatorname{Tr}\phi - \phi_{ii}) - \phi_{jj}(\operatorname{Tr}\phi - \phi_{jj}) + \frac{\operatorname{Tr}\rho(\phi)}{2(n-1)} = 0.$$

Suppose that ϕ has three distinct eigenvalues ϕ_{11}, ϕ_{22}, and ϕ_{33}. For any fixed i, the above equation is a quadratic in ϕ_{jj}, so we must have that the other two eigenvalues solve the quadratic specified by the third eigenvalue. So

$$\begin{aligned} \phi_{22} + \phi_{33} &= \operatorname{Tr}_\phi - (n-2)\phi_{11} \\ \phi_{11} + \phi_{33} &= \operatorname{Tr}_\phi - (n-2)\phi_{22}. \end{aligned}$$

Then $\phi_{11} = \phi_{22}$, a contradiction.

So ϕ has two distinct eigenvalues α and β. If both are repeated, then the quadratic necessitates that

$$\alpha + \beta = \operatorname{Tr}\phi - (n-2)\alpha = \operatorname{Tr}\phi - (n-2)\beta \implies \alpha = \beta,$$

a contradiction. So only one of the eigenvalues can have multiplicity. Without loss of generality, suppose $\phi = \alpha \operatorname{Id}_{n-1} \oplus \beta \operatorname{Id}_1$. Let ψ be a rank one form with $\psi_{nn} = \beta - \alpha$ and all other entries zero. Then $\phi = \alpha g + \psi$ as claimed. □

4 Open Questions

One interesting question would be to extend the results of this paper to indefinite metrics g.

Do Einstein canonical tensors span the space of Einstein tensors? Lemma 9 suggests that this could be answered by a construction realizing all Weyl tensors are linear combinations of Einstein canonical tensors.

Proposition 10 characterizes when canonical tensors are in the image of σ. When are sums or differences of canonical tensors in $\operatorname{Im}\sigma$? Since all tensors in the image are Kulkarni-Nomizu products, they can be written as a sum or difference of two canonical tensors. What can we then infer about these tensors?

What about taking multiple metrics g? Can these metrics be appropriately chosen to classify algebraic curvature tensors based on canonical tensor decompositions? In particular, choosing $\dim S^2(V^*)$ such metrics g_i guarantees by polarization that $\oplus \operatorname{Im}\sigma_i$ spans $\mathcal{A}(V)$. Can this be done with fewer metrics, and what properties of the Ricci-Weyl decomposition can help tackle this question?

Finally, a new result by Favazza [3] shows that the set of canonical tensors is dense in $\mathcal{A}(V)$ when $n = 3$. Is $\rho(S^2(V^*))$ dense in $S^2(V^*)$ in higher dimensions? Additionally, what can this image look like? In particular, consider $\rho(\mathbb{D}^{\dim S^2(V^*)})$, the image of the unit disk. This set is compact and connected; what can it look like?

Let π_W denote orthogonal projection onto $\mathcal{W}(V)$, the space of Weyl tensors. We can see that $\mathcal{W}(V) = \operatorname{span}\{\pi_W R_\phi \mid \phi \in S^2(V^*), \operatorname{Tr}\phi = 0\}$ as $\pi_W R_\phi = \pi_W R_{\phi+cg}, c \in \mathbb{R}$. Further, these traceless forms are relatively "nice"; for example, it is straightforward to see that the map ρ is at worst two-to-one on traceless forms (of rank at least three). What results can we obtain about linear combinations of canonical tensors of traceless forms?

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