Symmetries of Augmented Links: On FALs, Chain Links, and their FAL-Equivalent Counterparts

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Abstract. In this paper, we address symmetry groups of augmented links. We will focus on FALs and related links with a complement homeomorphic to that of an FAL. Our main focus will be on chain links which are flat FALs and links FAL-equivalent to chain links. Our overarching goal is to examine the relationship between the symmetry groups of links and links FAL-equivalent to them along with how disparate those two symmetry groups may be.
Keywords: symmetry group, augmented link, fully augmented link, crushtacean, FAL-equivalent, flat FAL, homeomorphic link complements, Dehn twists

1 Introduction

In this paper, we address the symmetry groups of augmented links. A symmetry group of a link L is the algebraic group of self-homeomorphisms of $(\mathbb{S}^3, L)$ up to isotopy. We denote the full symmetry group by $Sym(L)$ and the subgroup that preserves orientation by $Sym^+(L)$. Further, we denote the set of orientation-reversing symmetries on L by $Sym^-(L)$. Each symmetry of a complement of a hyperbolic link $M = \mathbb{S}^3 \setminus L$ (which all of the links discussed in this paper are) induces a symmetry of the link, so $Sym(L) \leq Sym(M)$ where $Sym(M)$. Our motivating question in this research is how different these two groups may be and how symmetries on M affect those on L. We begin by defining and introducing key terminology and verbiage.

Augmented links are a class of links constructed by augmenting a link L with a crossing circle around twist regions of L. *Fully augmented links (FALs)* are a subclass of augmented links referring to those where twist regions have exactly two strands and full twists are removed from within each crossing circle so each crossing circle contains (i.e. wraps around) zero crossings or one crossing. We call an FAL *flat* if all of its crossing circles contain zero crossings after full twists have been removed (See Figure 1).

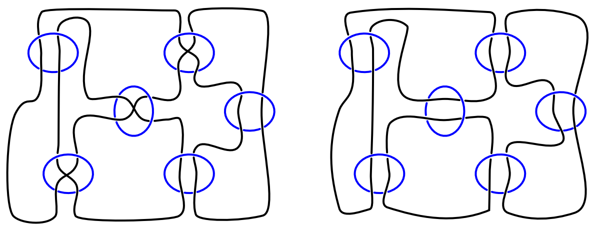


Figure 1: Example of an FAL on the left and a flat FAL on the right

A *crushtacean* of an FAL F is the dual of the painted triangulation of its associated ideal polyhedra and is denoted $C(F)$. (See Figure 2 for the crushtacean of the two FALs from Figure 1.) Painted edges of $C(F)$ correspond to crossing circles and non-painted edges corresponds to knot circles or strands (strands in the FAL that originate from the original link L). For more information and a more detailed explanation of FALs and crushtaceans, consult [5] and [3].

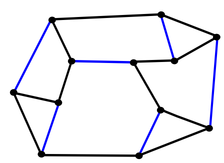


Figure 2: Painted Crushtacean corresponding to the FALs in Fig. 1

Chain links (CLs) are a specific type of flat FALs whose crushtaceans are prism graphs and have the same number of crossing and knot circles (See Figure 3). We will denote the n-chain link by P_n and the n-prism graph by Y_n.

If $k \in \mathbb{Z}$ then we define a *k-Dehn twist on a crossing circle X of an FAL L* to be the addition of k full Dehn twists within X (See Figure 4). We say that two links L_1 and L_2 are *FAL-equivalent* if we can perform a finite number of k-Dehn twists on L_1 to produce L_2 and we write $L_1 \sim_{DS} L_2$. Furthermore, in this case, we know that L_1 and L_2 have homeomorphic complements, so their symmetry groups are both subgroups of the common symmetry group of their complements.

A *projection plane of an FAL* is a plane containing the knot circles. Usually the choice of knot circles is non-arbitrary; however, in the case of chain links, the choice is not obvious because crossing circles can be transformed into knot circles. Therefore, chain links have two projection planes, which we will denote ρ_1 and ρ_2. Figure 5 features a diagram pointing out those two planes.

Now we have defined the necessary terminology pertaining to the symmetry groups of the augmented links in question, so we will proceed to use topological,

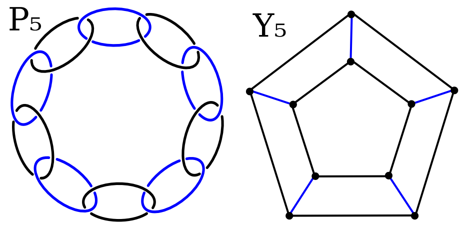


Figure 3: P_5 link with crossing circles in blue and its corresponding painted crushtacean Y_5

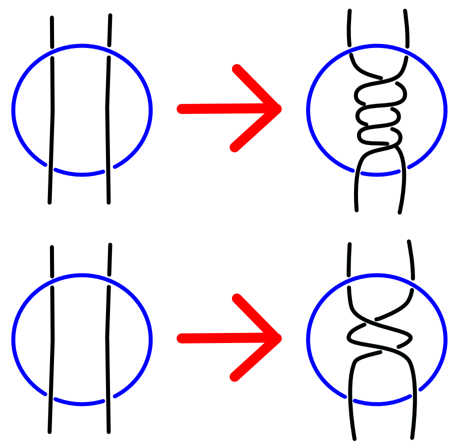


Figure 4: A +2-Dehn twist (top) and a -1-Dehn twist (bottom)

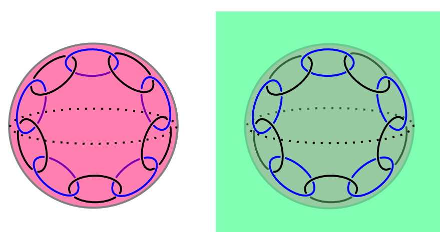


Figure 5: ρ_1 is the sphere shaded in pink (left) ρ_2 is the plane shaded in green (right)

algebraic, and graph theoretic techniques to determine useful results for symmetry groups. First, we address chain links and links that are FAL-equivalent to them.

Remark. Moving forward, we denote $\{e\}$ the trivial group by E.

2 Symmetries of Chain Links

In this section, we address the symmetry groups of chain links P_n and links that are FAL-equivalent to P_n. By [4], we know that P_n admit no hidden symmetries either orientation-preserving or reversing and that $Sym(P_n) \cong Sym(M_n)$ (as well as that $Sym^+(P_n) \cong Sym^+(M_n)$) where M_n is the manifold that is the complement of P_n in $\mathbb{S}^3$ and $n \geq 4$. For our purposes, we will denote crossing circles by C_i where $1 \leq i \leq n$ where we choose an arbitrary circle from the link to correspond to C_1 and proceed to label the crossing circles in order clockwise. We do the same with knot circles but denote them with K_i. Further, we let $D(C_i)$ to be the number of full Dehn twists a crossing circle C_i contains (for example, $D(X) = +2$ for the crossing circle in the top-right of Figure 5). A *crossing disk* is a disk for which a crossing circle forms the boundary and similarly for a *knot disk* (we will use C_i and K_i to refer to these disks as well as the link components). We will choose a point within the open crossing disk C_i and in the projection plane ρ_1 and label it p_i. Similarly, we choose a within the open knot disk K_i and label it p_i'. (See Figure 6). The choices for these points do not matter because we are concerned with homeomorphisms and these two different arbitrary choices for a p_i or p_i' are homeomorphic.

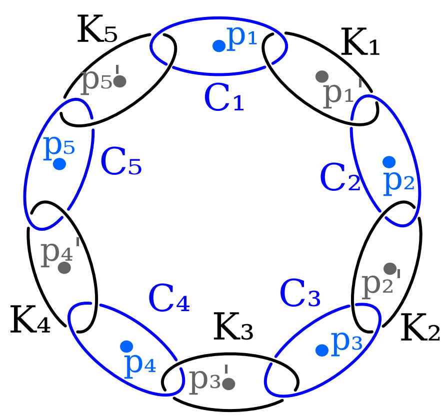


Figure 6: P_5 link with our chosen labelling

Theorem 2.1 (Theorem 6.1 from [4]) If we denote α to be the inward rotation by $\frac{\pi}{2}$ around the circular axis of symmetry followed by a rotation $\frac{\pi}{n}$ (effectively changing a crossing circle to a knot circle and vice versa), β to be the inward rotation by π around the circular axis of symmetry, and γ to be the π rotation around the axis of symmetry of the link in the projection plane containing our designated knot circles, then the set $\{\alpha, \beta, \gamma\}$ are generators for $Sym^+(P_n)$ when $n \geq 4$. (See Figure 7).

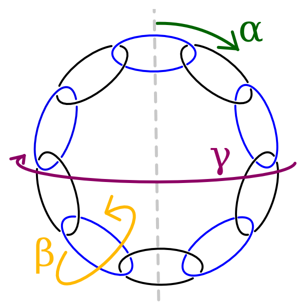


Figure 7: P_5 link with generators for orientation-preserving symmetry group

Remark. For all our work in this section forwards, we assume $n \geq 4$.

Proposition 2.1 If n is odd then $|\alpha| = 4n$, and if n is even then $|\alpha| = 2n$.

Proof. If n is even and we allow α^{2n} to act on $(\mathbb{S}^3, P_n)$ then all the C_i are permuted to their original positions, and we obtain the identity. Therefore, the order of α must be $2n$.
However, if n is odd, then allowing α^{2n} to act on $(\mathbb{S}^3, P_n)$ is not equivalent to the identity but a homeomorphism equivalent to allowing β to act on $(\mathbb{S}^3, P_n)$, i.e. $\alpha^{2n} = \beta$. The order of β is clearly 2 and so the order of α in this case is $4n$. ☐

Theorem 2.2 If n is even, then $Sym^+(P_n) \cong D_{2n} \times \mathbb{Z}_2$, and if n is odd, then $Sym^+(P_n) \cong D_n \times \mathbb{Z}_4$.

Proof. Case 1. Suppose n is even.
Then by Theorem 2.1 and Proposition 2.1, all the symmetries generated by α and γ form a subgroup of $Sym^+(P_n)$ of index 2 that can be induced by and in turn induce the symmetries on the $2n$-gon connecting the p_i and p_i'. Therefore this subgroup is isomorphic to D_{2n}, and $\langle \alpha, \gamma \rangle \lhd Sym^+(P_n)$. Let B be the subgroup generated by β. Then clearly $\mathbb{Z}_2 \cong B \lhd Sym^+(P_n)$, i.e. B is normal in $Sym^+(P_n)$. And so by applying the Recognition Theorem of Direct Products,

we obtain that $Sym^+(P_n) \cong D_{2n} \times \mathbb{Z}_2$.

Case 2. Now suppose n is odd.
By Proposition 2.1, we can generate all of $Sym^+(P_n)$ with only α and γ. by Proposition 2.1. Let A be the subgroup of $Sym^+(P_n)$ generated by α. It is trivial to show that $A \cong \mathbb{Z}_{4n}$ and so A is index 2 in $Sym^+(P_n)$ and $A \lhd Sym^+(P_n)$. Because n and 4 are relatively prime, then $\mathbb{Z}_{4n} \cong \mathbb{Z}_n \times \mathbb{Z}_4$. Let G be the subgroup of $Sym^+(P_n)$ generated by γ which is clearly order 2.
Take $\phi_\sigma : G \to Aut(A)$ be defined as left conjugation, i.e. $\phi_\sigma(x) = \sigma x \sigma^{-1}$ and $\sigma \in G$. Note that $\phi_e = id$ and $\phi_\gamma(x) = \gamma x \gamma = x^{-1}$. hen by the Recognition Theorem for Semidirect Products (Theorem DF.5.12 from [1]), we know that $Sym^+(P_n) \cong (\mathbb{Z}_n \times \mathbb{Z}_4) \rtimes_\phi G \cong \mathbb{Z}_4 \times (\mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2) \cong D_n \times \mathbb{Z}_4$. □

Now that we have identified $Sym^+(P_n)$, the natural next step is to identify $Sym(P_n)$ the full symmetry group up to isomorphism. To this end, we will largely rely on group theoretic techniques and the above remark.

Theorem 2.3 If n is even then $Sym(P_n) \cong D_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and if n is odd then $Sym(P_n) \cong D_n \times D_4$.

Proof. Let R be the order 2 subgroup of $Sym(P_n)$ generated by the reflection of P_n across ρ_1, which we will denote by r, i.e. $R = \langle r \rangle$. Because $Sym^+(P_n)$ is index 2 in $Sym(P_n)$, then $Sym^+(P_n) \lhd Sym(P_n)$. By definition of $Sym^+(P_n)$, we know that $R \cap Sym^+(P_n) = E = \{e\}$.
Take $\phi_\sigma : R \to Aut(Sym^+(P_n))$ be defined as left conjugation, i.e. $\phi_\sigma(x) = \sigma x \sigma^{-1}$ where $\sigma \in R$. Note that $\phi_e = id$ and $\phi_r(x) = rxr = x^{-1}$. By the Recognition Theorem for Semidirect Products, we know that $Sym(P_n) \cong Sym^+(P_n) \rtimes_\phi R$.
Case 1. Suppose n is even.
Then $Sym(P_n) \cong Sym^+(P_n) \rtimes_\phi R \cong (D_{2n} \times \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}_2$. By associativity of groups, we can move the parentheses to obtain $Sym(P_n) \cong D_{2n} \times (\mathbb{Z}_2 \rtimes_\phi \mathbb{Z}_2)$. Note that for any ϕ, it is the case that $\mathbb{Z}_2 \rtimes_\phi \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so $Sym(P_n) \cong D_{2n} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
Case 2. Otherwise, suppose n is odd.
Then $Sym(P_n) \cong Sym^+(P_n) \rtimes_\phi R \cong (D_n \times \mathbb{Z}_4) \rtimes_\phi R \cong D_n \times (\mathbb{Z}_4 \rtimes_\phi R)$. Because $\mathbb{Z}_4 \rtimes_\phi \mathbb{Z}_2 \cong D_4$, then $Sym(P_n) \cong D_n \times D_4$, as desired. □

Now that we have determined the full symmetry group of P_n, we will address links L such that $L \sim_{DS} P_n$. We know that the symmetries of these links will be a subgroup of that of M_n, where $M_n = \mathbb{S}^3 \setminus P_n$. Further, it has been shown that $Sym(M_n) \cong Sym(P_n)$ and $Sym^+(M_n) \cong Sym^+(P_n)$ [4]. Table 2.1 below lists all the possible subgroups of D_5, D_4, D_{12}, and $\mathbb{Z}_2$. Because we know $Sym(P_5) \cong D_5 \times D_4$ and $Sym(P_6) \cong D_{12} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we can use these subgroups from the table to determine all the subgroups of $Sym(P_5)$ and $Sym(P_6)$.

Subgroups of D_5	Subgroups of D_4	Subgroups of D_{12}	Subgroups of $\mathbb{Z}_2$
E	E	E	E
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_5$	$\mathbb{Z}_4$	$\mathbb{Z}_{12}$	
D_5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$	
	D_4	$\mathbb{Z}_4$	
		$\mathbb{Z}_3$	
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	
		D_6	
		D_4	
		D_3	
		D_{12}	

Table 2.1 Lists the subgroups of D_5, D_4, D_{12}, and $\mathbb{Z}_2$, from which we can use direct products to construct all the subgroups of $D_4 \times D_5$ and $D_{12} \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which are the full symmetry groups of P_5 and P_6, respectively.

We move forward by using Dehn twists to construct links whose complements are homeomorphic to M_n but have fewer symmetries themselves than P_n. We wish to determine exactly which subgroups of $Sym(P_n)$ the symmetry groups of links constructed in this fashion will have. Towards this goal, we will characterize all the possible options for $Sym^+(L)$ where $L \sim_{DS} P_n$.

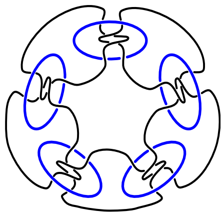
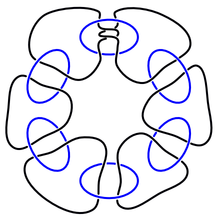
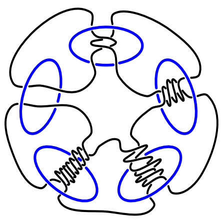
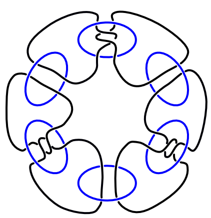
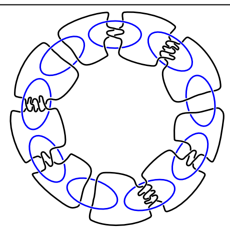
Link Diagram	$Sym^+(L)$	Lemma with a General Construction
	$D_5 \times \mathbb{Z}_2$	Lemma 2.2
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Lemma 2.3
	$\mathbb{Z}_2$	Lemma 2.4
	$D_3 \times \mathbb{Z}_2$	Lemma 2.5
	$\mathbb{Z}_3 \times \mathbb{Z}_2$	Lemma 2.6

Table 2.2 Possible links FAL-equivalent to P_5, P_6, and P_9.

PROPOSITION 2.2 If $L \sim_{DS} P_n$ and there is some $\sigma \in Sym(L)$ where σ acts on L to take some C_i to some K_j then $L = P_n$.

Proof. It is clear that any symmetry that takes a crossing circle to a knot circle must take all the crossing circles to knot circles and vice versa. Because σ must preserve linking numbers of pairs of components, we know that the linking numbers of any two adjacent knot circles K_i and K_{i+1} must be zero. This means L is exactly P_n. $\square$

Proposition 2.2 can be interpreted as saying that, for any $L \sim_{DS} P_n$ where $L \neq P_n$, symmetries of L can only act on L in such a way that crossing circles are strictly permuted to other crossing circles.

PROPOSITION 2.3 If $L \sim_{DS} P_n$ and $\sigma \in Sym(L)$ where σ is a rotational symmetry and $\sigma(C_i) = C_j$ with $i \neq j$ and $gcd(j-i, n) = 1$, then there is $k \in \mathbb{Z}$ so that $D(C_i) = k$ for all $1 \leq i \leq n$.

Proof. We know $(j-i)x + ny = 1$ has integer solutions for x and y. Then if we allow σ^x to act on $(\mathbb{S}^3, L)$, we will obtain a rotational symmetry that takes C_i to C_{i+1}. Therefore, the signed linking number between knot circles must be invariant at each crossing circle as desired. $\square$

LEMMA 2.1 If $L \sim_{DS} F$ where F is an FAL with the same painted crushtacean as P_n, then $Sym^+(L) \neq E$.

Proof. Note that β preserves Dehn-twist-numbers both in magnitude and sign, so $\beta \in Sym^+(L)$. Because β is a non-trivial element of order 2, we know $Sym^+(L) \neq E$. $\square$

By Lemma 2.1, we already know that not all subgroups of $Sym(P_n)$ can be achieved by breaking symmetries via full Dehn twists. It remains to determine the fates of the remaining possible subgroups. Further note that the hypothesis of this lemma is less restrictive than $L \sim_{DS} P_n$ and also applies to links with half twists when full Dehn twists are removed.
Now we would like to construct a general case of first link from Table 2.2 $L \sim_{DS} P_n$ with $Sym^+(L) \cong D_n \times \mathbb{Z}_2$.

LEMMA 2.2 If $L \sim_{DS} P_n$ with $m \neq 0$ full Dehn twists at each crossing circle, i.e. $D(C_i) = m$ for all i, then $Sym(L) \cong Sym^+(L) \cong D_n \times \mathbb{Z}_2$.

Proof. The addition of the same signed number of full Dehn twists to P_n will preserve the rotational symmetry on the n-gon connecting the p_i, γ, and β. So we can use the n-gon of the p_i to induce $\langle \alpha', \gamma \rangle$ where α' is the rotation of $\frac{2\pi}{n}$ about the central point of the n-gon. So $D_n \times \mathbb{Z}_2 \cong \langle \alpha', \gamma \rangle \leq Sym(L)$.
We now show that every element of $Sym(L)$ is in $\langle \alpha', \gamma \rangle$. Let σ be a symmetry of L then each C_i must be permuted to itself or another crossing circle by Proposition 2.2. Without loss of generality, suppose that σ takes C_1 to C_j. If

σ preserves orientation of the knot circles in the projection plane then σ clearly must be a either a rotation, β, or γ. We know that σ must preserve orientation because a reflection would change the signs on the $D(C_i)$. It is trivial to show that $|Sym(L)| = \frac{|Sym(P_n)|}{2 \cdot 2} = \frac{16n}{4} = 4n$, and we have that $|\langle \alpha', \gamma \rangle| = 4n$, so it must be the full symmetry group. $\square$

Now we consider the second link L_2 from Table 2.2. In this case, reflections as a symmetry are broken along with rotational symmetries inherited from α other than β, but γ remains a symmetry of $(\mathbb{S}^3, L_2)$. We would like to construct a general link $L \sim_{DS} P_n$ that breaks the symmetries in the same way as described for L_2 so that $Sym^+(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 2.3 If $L \sim_{DS} P_n$ and there is $j \in \{1, \ldots, n\}$ such that $D(C_j) = k$ and $D(C_i) = m \neq k$ for all $i \neq j$, then $Sym(L) \cong Sym^+(L) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Because symmetries of L must preserves $|D(C_i)|$ and we are assuming $n \geq 4$, any symmetry of L must take C_j to itself. From the symmetries of P_n, this leaves β, γ, and reflection across ρ_1. Both β and γ are valid symmetries, but the action of reflection across ρ_1 results in L' not homeomorphic to L since reflection changes the sign of a Dehn twist. Therefore $Sym^+(L) \cong Sym(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ $\square$

By Lemma 2.1, we know that $Sym^+(L)$ will always have a subgroup isomorphic to $\mathbb{Z}_2$. In the case of the third link L_3 from Table 2.2, we know there are cases of $L \sim_{DS} P_n$ with $Sym^+(L_3) \cong \mathbb{Z}_2$. The following lemma will give a more general case of constructions that eliminate all symmetries but β though it is by no means comprehensive of all such constructions.

Lemma 2.4 Suppose $a_1, a_2, \ldots, a_n \in \mathbb{N}$ where $a_i \neq a_j$ for all $i \neq j$. If $L \sim_{DS} P_n$ where $D(C_i) = a_i$ for all i, then $Sym(L) \cong Sym^+(L) \cong \mathbb{Z}_2$.

Proof. By Lemma 2.2, we know $\mathbb{Z}_2 \leq Sym(L)$. Suppose $\sigma \in Sym(L)$. Then σ preserves the absolute value of linking numbers between the knot circle components, which are unique at each crossing circle by construction, therefore each C_i must go to C_i. Reflections in the projection plane change the signs on the $D(C_i)$ and so cannot be a symmetry of L. Hence, the only nontrivial symmetry remaining is β, the inner twist. $\square$

Now consider the fourth link from Table 2.2. In this case only some rotational symmetry inherited from P_n is lost in such a way that we can induce those symmetries along with γ by a triangle connecting p_2, p_4, and p_6. Note that this works precisely because $3|6$. We proceed to construct a more general case of this kind of symmetry group on L.

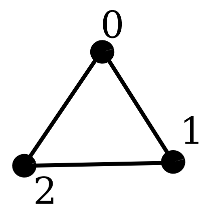


Figure 8: This is the symmetry graph of $0, 1, 2$ and $\widehat{Aut}(G) \cong E$

LEMMA 2.5 Suppose $k|n$ with $k \geq 2$, and $L \sim_{DS} P_n$ where for all $i \in \{1, 2, \ldots, \frac{n}{k}\}$, we have $D(C_{ki}) = m$ and $D(C_j) = l \neq m$ for all other j where $k \nmid j$. Then $Sym^+(L) \cong D_k \times \mathbb{Z}_2$.

Proof. Note that because symmetries preserve the absolute value of linking number, we know that any symmetry of L must take C_k to some C_{ki}. Otherwise, the proof for this follows the same logic as for Lemma 2.3 but we limit ourselves to the k-gon constructed by connecting the $p_{\frac{n}{k}i}$. □

Consider the final link L_5 from Table 2.2. This link clearly has a symmetry of rotation $\frac{2\pi}{3}$ about a central point as well as β. However, note that $\gamma \notin Sym(L_5)$ because the crossing numbers form the sequence $0, 1, 2, 0, 1, 2, 0, 1, 2$ which is an asymmetric sub-sequence $0, 1, 2$ repeated thrice, so $Sym^+(L_4) \cong \mathbb{Z}_3 \times \mathbb{Z}_2$. We would like to construct an even more general kind of asymmetry that will cover all $L \sim_{DS} P_n$ with this particular kind of symmetry group where γ is not a possible symmetry but L does have a non-trivial rotational symmetry about a central point. This motivates the following definition.

DEFINITION 2.1 Let $a_1, a_2, \ldots, a_m \in \mathbb{Z}$ and consider the sequence $\mathbf{a} = \langle a_1, a_2, \ldots, a_m \rangle$. Let G be the cycle graph on m vertices (with $V = p_1, \ldots, p_m$ and $E = \{(p_i, p_j) : |i - j| = 1 \ mod \ m\}$) where the vertices are decorated with the values of the a_i. We will call $\mathbf{a}$ *symmetry-reducing* if the automorphism group of G preserving values of vertices is the trivial group. Moreover, we shall refer to G as the symmetry graph of $\mathbf{a}$ and the automorphism group preserving values at vertices as $\widehat{Aut}(G)$.

LEMMA 2.6 Suppose $m|n$, $m \geq 3$, and let $\mathbf{a} = \langle a_1, \ldots, a_m \rangle \in \mathbb{Z}^m$ such that $\mathbf{a}$ is symmetry-reducing. Let $L \sim_{DS} P_n$ and $D(C_i) = a_m$ if $m|i$. Otherwise if $m \nmid i$ then take $k = i \ mod \ m$ and let $D(C_i) = a_k$. Then $Sym^+(L) \cong \mathbb{Z}_m \times \mathbb{Z}_2$.

Proof. The proof here is similar to that of Lemma 2.5 but this particular construction eliminates the γ-symmetry inherited from P_n. □

After compiling these particular generalized symmetry groups of $L \sim_{DS} P_n$ from Lemmas 2.2-2.6, the natural question is: is this a complete list of symmetry groups? The following theorem addresses this question.

THEOREM 2.4 Let $L \sim_{DS} P_n$ then $Sym^+(L)$ is one of the following:

1. $\mathbb{Z}_2$
2. $\mathbb{Z}_2 \times \mathbb{Z}_2$
3. $D_n \times \mathbb{Z}_4$ if n odd
4. $D_{2n} \times \mathbb{Z}_2$ if n even
5. $D_k \times \mathbb{Z}_2$ where $k|n$
6. $\mathbb{Z}_k \times \mathbb{Z}_2$ where $k|n$ and $\frac{n}{k} \geq 3$

Proof. Suppose the orientation-preserving symmetry group of L is not (2) through (6). By Lemmas 2.2-6, we know that L does not follow any of the constructions from those lemmas. By Lemma 2.1, we know $\beta \in Sym^+(L)$ so there is at least a subgroup of $Sym^+(L)$ that is isomorphic to $\mathbb{Z}_2$.
Let σ be a nontrivial orientation-preserving symmetry of L. The only orientation-preserving symmetry of L that takes every C_i to C_i is β.
So suppose, for a contradiction and without loss of generality, σ takes C_1 to C_j for some $j \neq 1$. Then because symmetries preserve the Dehn twists, we know $D(C_1) = D(C_j)$. If $gcd(j-1, n) \neq 1$ then this induces a case from Lemmas 2.5-6. So $gcd(j-1, n) = 1$ but then by Proposition 2.1, we have the construction from Lemma 2.2, a contradiction. □

Now that we have fully classified the orientation-preserving symmetry groups for $L \sim_{DS} P_n$, it remains to determine the full symmetry groups. We can address this problem simply because Lemmas 2.2-2.4 address the full symmetry groups along with the orientation-preserving subgroup so that leaves adding reflections to the orientation-preserving symmetry groups for constructions from Lemmas 2.5-6 and links that fall outside Lemmas 2.2-2.6, for which the symmetry group is isomorphic to $\mathbb{Z}_2$. So the corollary below naturally follows from Theorem 2.4.

Corollary 2.1 Let $L \sim_{DS} P_n$ then $Sym(L)$ is one of the following:

1. $\mathbb{Z}_2$
2. $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \rtimes_\phi \mathbb{Z}_2$
3. $D_n \times \mathbb{Z}_4$ if n odd
4. $D_{2n} \times \mathbb{Z}_2$ if n even
5. $D_k \times \mathbb{Z}_2$ where $k|n$
6. $(D_k \times \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}_2 \cong D_k \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where $k|n$
7. $\mathbb{Z}_k \times \mathbb{Z}_2$ where $k|n$ and $\frac{n}{k} \geq 3$
8. $(\mathbb{Z}_k \times \mathbb{Z}_2) \rtimes_\phi \mathbb{Z}_2 \cong \mathbb{Z}_k \times \mathbb{Z}_2 \times \mathbb{Z}_2$ where $k|n$ and $\frac{n}{k} \geq 3$

And so we have categorized all the symmetries of augmented links that are FAL-equivalent to P_n.

3 Symmetries of the Crushtacean for FALs

Consider a chain link P_n and its crushtacean Y_n with the edges connecting the inner and outer n-gons of Y_n painted. Note that $Aut'(Y_n) \cong D_n \times \mathbb{Z}_2$ where we take $Aut'(G)$ to be the automorphism group of a painted graph G that preserves painting. The generators of $Aut'(Y_n)$ are α', β', and γ': where α' is a $\frac{2\pi}{n}$-rotation; β', lifting the middle n-gon of Y_n to the outside; and γ', reflection across the axis of symmetry in the plane. Based on our work from Theorem 2.2, we can construct an injection $h : Aut'(Y_n) \to Sym^+(P_n)$ where $h(\alpha') = \alpha^2$, $h(\beta') = \beta$, and $h(\gamma') = \gamma$. Therefore, we know in the case of chain links that $Aut'(Y_n) \leq Sym^+(P_n)$. One naturally wonders if such a relationship can be extended more generally to FALs, especially as it has been shown that $Aut'(C)$ injects to the symmetry group of the complement [3]. To answer this question, we require several tools from graph theory and geometry.

Definition 3.1 A *polytope* is geometric object with flat faces.
Definition 3.2 A *skeleton of a polytope* is a graph whose vertices are the vertices of the polytope with edges connecting vertices that are connected by an edge in the polytope. For example, the skeleton of a pentagonal prism would be Y_5.

We will need to invoke a result from [2] about a particular property of crushtaceans of FALs that then allows us to borrow a convenient theorem from graph theory for our results. These two theorems are listed below.

Theorem 3.1. [2] Crushtaceans of FALs are 3-connected.
Theorem 3.2 [6] For every tri-connected graph G, there exists a polytope P in $\mathbb{R}^3$ such that G is the skeleton of P and P displays the symmetries of G.

Now we have the necessary tools to show a strong relation between $Sym(L)$ and $Aut'(C)$.

Theorem 3.3. If L is a flat fully augmented link with painted crushtacean C then there exists an injection $h : Aut'(C) \to Sym^+(L)$.

Proof. By Theorem 3.1, we know that C is triconnected and so we may construct a polytope P that displays the symmetries of C using Theorem 2.2. Now if we paint the edges of P so that it corresponds to the painting on C, we can then use a method analogous to Purcell's [An Intro to FALs] method to reconstruct L on P by wrapping crossing circles around painted edges and the knot circles are reconstructed on the surface of P by following the unpainted edges. (See Figure 10 for an example construction). Therefore, each symmetry of P induces a symmetry of L, and so any symmetry of C will induce one on L. □

Corollary 3.1 If L is flat FAL with painted crushtacean C then $Aut'(C) \rtimes \mathbb{Z}_2 \leq Sym(L)$.

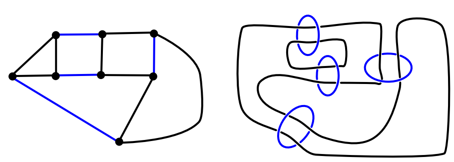


Figure 9: Example of a crushtacean and recovering the flat FAL corresponding to it

Though this result is very interesting, we are restricted to flat FALs. However, we will extend this result to all FALs by invoking another of Twogood's theorems included below for reference.

Theorem 3.4 [2] Up to flype-equivalence, a signed, painted crushtacean corresponds exactly to one FAL.

We will denote $Aut''(G)$ to be the subgroup of $Aut(G)$ where elements preserve painting and signing on painted edges. Then we can create a relationship

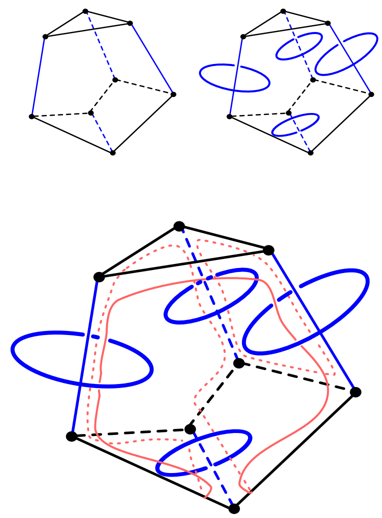


Figure 10: Going clockwise: (1) Conversion of the crushtacean from Fig. 10 to a painted polytope (2) Adding crossing circles to painted edges (3) Drawing in knot circles from the unpainted edges

extending to all FALs that is a generalization of Theorem 2.3.

> COROLLARY 3.1 If L is an FAL with signed painted crushtacean C then there exists an injection $h : Aut''(C) \to Sym(L)$.

Proof. The proof here is the exact same as for Theorem 2.3 but adding the condition of signs on the painted edges to account for half-twists. ☐

Based on our own results and previous work, we think that there are more restrictive symmetry results to be found for FALs that invoke more sophisticated methods from geometric topology on the complements of FALs. We list the following conjecture that we believe is provable using such techniques.

> CONJECTURE 3.1 Suppose L is a b-prime flat FAL with crushtacean C and $M = \mathbb{S}^3 \setminus L$ arithmetic. Then $Sym^+(L) \cong Aut'(C)$.

For information on b-prime FALs, please consult [8].

4 Conclusion and Future Directions

Based on our work in Section 2, we know that some symmetries of the complement are inherited by links FAL-equivalent to P_n. So for this specific class of links, we know that not all subgroups $H \leq Sym(P_n) \cong Sym(M_n)$ have some $L \sim_{DS} P_n$ so that $Sym(L) \cong H$. The natural question is whether or not this extends past FAL-equivalent links for P_n. A natural next step in this research would be to examine nested links, for which there is a natural way to use a special painting on n-prism graphs to construct links with complements homeomorphic to M_n. Classifying the symmetry groups of this additional class of links would be a useful next step in determining a stronger and more useful generalization of how the symmetry group of the complement limits that of the link itself.

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