Classifying Cuboctahedral FALs

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Abstract

We give a combinatorial and geometric description for an infinite class of cuboctahedral FALs that are distinct from other known classes. Our construction is not possible on octahedral FALs, as shown by Purcell., so it highlights the distinction between these two classes of arithmetic FALs. We also utilize the work of Worrell, specifically the previously unclassified example presented in his work as a basis for this classification.

1 Background

It has recently been proven by Hoffman and Worden that octahedral and cuboctahedral FALs make up all arithmetic FALs. As it stands, octahedral FALs have been fully classified by Purcell, but the same has yet to have been completed for cuboctahedral FALs[2]. In his work, Worrell presents one method for creating cuboctahedral FALs, however, this process does not contain all valid cuboctahedral FALs. The FALs produced from this process belong to a class referred to as the 1st Family [3]. We will present a new construction that produces cuboctahedral FALs that strictly do not belong to the 1st family.

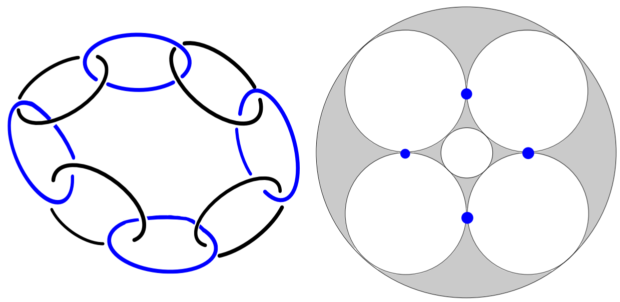


Figure 1: 4-Link Chain to it's circle packing

1.1 Fully Augmented Links

We will begin by defining the basics of FALs and the associated graphs we utilize in this paper.

Definition 1.1. (Fully Augmented Link) Given a link diagram, take any twist region and place a crossing circle around the strands. Now, reduce the number of twists contained in each crossing circle modulo 2. The resulting link is called a fully augmented link (FAL).

We will consider the circle packing of a cuboctahedral FAL, specifically the 4-link chain. The circle packing, as seen in Figure 1, represents the right-angled ideal polyhedra of the polyhedral decomposition of Purcell's Proposition 2.2 [2]. We refer to the blue points of Figure 1 (right) as *crossing circle points*. This circle packing is used to generate graphs that allow us to further study FALs.

Definition 1.2. (Circle Packing) The circle packing of an FAL is an arrangement of non-overlapping circles where shaded regions correspond to crossing disks and unshaded regions correspond to the projection plane. From the circle packing, we can generate two planar graphs.

Definition 1.3. (Crushtacean) The crushtacean of a fully augmented link is the graph with vertices corresponding to shaded regions of the associated circle packing where two vertices are adjacent if the shaded regions are tangent to each other. Every vertex in a crushtacean is 3-valent.

Definition 1.4. (Nerve) The nerve of a FAL is the dual graph of the crushtacean. Equivalently, the nerve has vertices corresponding to unshaded regions of the circle packing with edges connecting vertices whose associated polygons are tangent to each other. Every region on a nerve is a triangle.

1.2 Geometric Background

We now shift to a geometric view of Cuboctahedral FALs. There is a significant amount of geometric background needed to fully understand the results of this paper. Specifically, the knowledge of cuboctahedral FAL complements. The complement of the FAL is where we see cuboctahedra and study their properties to better understand the cuboctahedral FAL.

Definition 1.5. (Hyperbolic Plane) A plane in the upper half-space model is either a hemisphere or a vertical half-plane.

(a) Regular Ideal Cuboctahedron (b) Regular Ideal 4-Antiprism

Figure 2: Hyperbolic Polyhedra

Definition 1.6. (Cuboctahedron) A **cuboctahedron** is a polyhedron with 6 square faces and 8 triangular faces. A cuboctahedron can be produced by truncating the vertices of a cube to form the triangles.

In hyperbolic space, we are able to study the unique regular ideal cuboctahedron. Hereafter when we refer to a cuboctahedron, we are referring to the regular ideal cuboctahedron.

Definition 1.7. (Regular Ideal Cuboctahedron) An ideal cuboctahedron is a cuboctahedron whose vertices are all ideal. A **regular ideal cuboctahedron** is an ideal cuboctahedron whose dihedral angles are all right angles (Figure 2a).

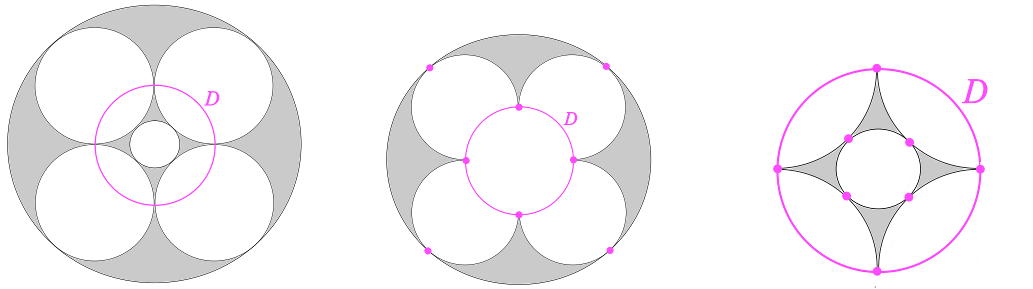
Definition 1.8. (Regular Ideal 4-Antiprism) A 4-Antiprism is a polyhedron having two square faces in parallel planes, with eight alternating-facing triangles connecting the square faces. A **Regular Ideal 4-Antiprism** is a 4-Antiprism whose vertices are all ideal and whose dihedral angles are all $\frac{\pi}{2}$ (Figure 2b).

Remark 1.9. Two 4-Antiprisms glue together along quadrilateral faces to form a cuboctahedron. Conversely, a cuboctahedron can be sliced with a plane to form two 4-Antiprisms. This can be seen in Figures 2a and 2b. In a circle packing, the plane that slices a cuboctahedron into two 4-Antiprisms is represented by the circle that intersects all crossing circle points, D, as seen in Figure 3a. Notice that outside of D in Figure 3b, we have one quadrilateral (the outermost circle), 4 shaded triangles, and 4 unshaded triangles. Including D as the second quadrilateral, we obtain a 4-Antiprism. Similarly, the interior of D seen in Figure 3c plus D gives the other 4-Antiprism.

Definition 1.10. (Multi-Polyhedral Points) A tangency point in a circle packing is Multi-Polyhedral if it is a point where two polyhedra meet (Figure 4a).

Remark 1.11. In Purcell's work with octahedra, we only allow for gluing along shaded faces. Note that conversely, we are working with cuboctahedra thus allowing for more space, making the gluing of cuboctahedra along unshaded faces possible.

Lemma 1.12. If two cuboctahedra A and B are glued along unshaded quadrilateral faces, then there exists a cuboctahedron made of half of A and half of B.

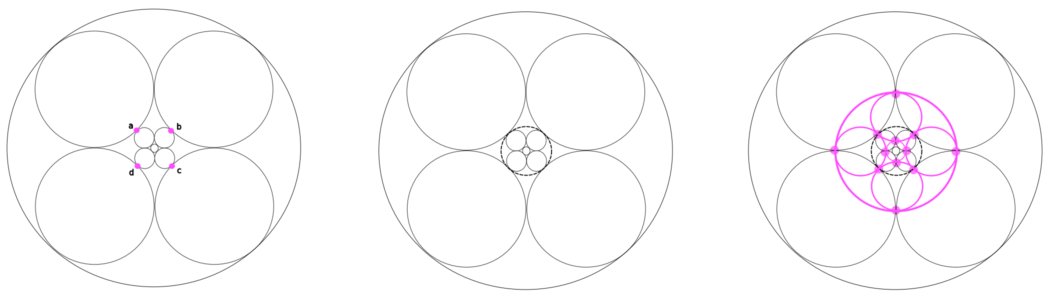


(a) The slicing of a cuboctahedron

(b) The outer 4-Antiprism from slicing

(c) The inner 4-antiprism from slicing

Figure 3: Slicing a Cuboctahedron into 4-Antiprisms



(a) Multi-Polyhedral Points

(b) Addition of one Cuboctahedron in P_+

(c) Cuboctahedron found by ETGS

Figure 4

Proof. We begin with a cuboctahedron, A, and consider its circle packing, C. Without loss of generality, consider the innermost circle, I, of C. By adding an isometric copy of C to I, as in Figure 4b, we glue a new cuboctahedron, B, to an unshaded face of A. Note that I is not present in the circle packing after the gluing because the bottom face of A is now interior to the polyhedron $A \cup B$.

We will now consider the plane that slices the cuboctahedra as in Remark 1.9. The plane through all crossing circle points in A produces one ideal quadrilateral, and the plane through all crossing circle points in B produces another. Connecting a crossing circle point from B, its adjacent multi-polyhedral points, and the cocircular crossing circle point from A, yields a plane. We can do this at each crossing circle point in B to obtain four more quadrilateral faces, as well as 8 triangular faces as in Figure 4c. Thus, we have found an ideal polyhedron, specifically a regular ideal cuboctahedron which we will name G, contained in the polyhedron that comes from gluing two cuboctahedrons.

Notice that the quadrilaterals of G are shaded, and the triangles of G are unshaded. Now consider the section of G that is contained in I. The bottom quadrilateral of G, as well as the bottoms of the triangles of G, are contained in I. Thus half of G is in B, leaving the other half of G in A. $\square$

1.3 The 1st Family

We will now introduce the family of cuboctahedral FALs introduced by Worrell that decompose in the same manner as octahedral FALs as described by Purcell [3] [2].

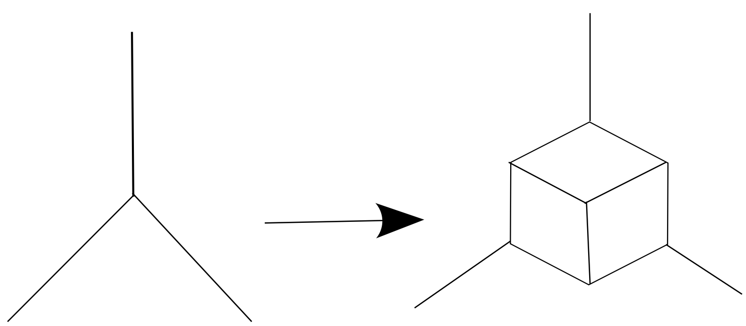


Figure 5: Vertex Hexagon Expansion

Definition 1.13. (Vertex Hexagon Expansion) To perform a vertex hexagon expansion on the crushtacean of an FAL, begin by taking any vertex and replace it with a hexagon. On every other vertex of the hexagon, draw an edge meeting at the center of the polygon, there should be 3. On every vertex that was skipped, draw an edge connecting to one of the vertices that was adjacent to the original vertex, such that each vertex from the hexagon is adjacent to a different vertex as describe in Figure 5.

In his work, Worrell defines the central triangular subdivision as well as proved that when performed on the nerve of a cuboctahedral FAL, it produces cuboctahedral FALs belonging to the first family. For our purposes, we will define the corresponding move for the crushtacean.

Definition 1.14. (1st Family Cubeoctahedral FALs) The 1st family of cuboctahedral FALs is all cuboctahedral FALs obtained by performing some number of vertex hexagon expansions on the simple cuboctahedral FAL.

2 A Combinatorial View of Cuboctahedra

Worrell presents a cuboctahedral FAL that cannot be created using the vertex hexagon expansion. We introduce a construction that can be used to not only get this example, but that can be used to create an infinite family of cuboctahedral FALs that do not belong to the 1st Family.

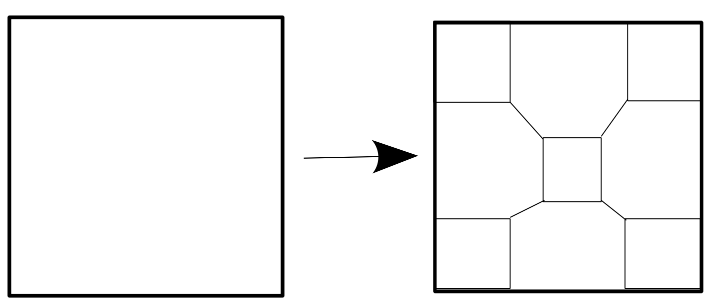
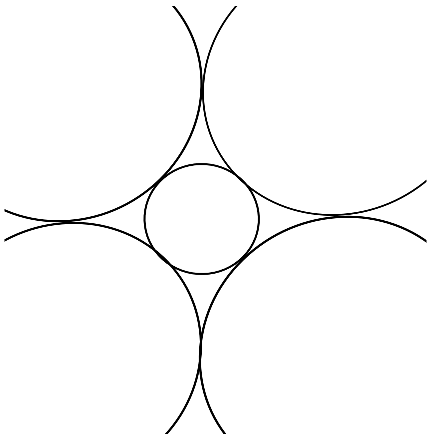
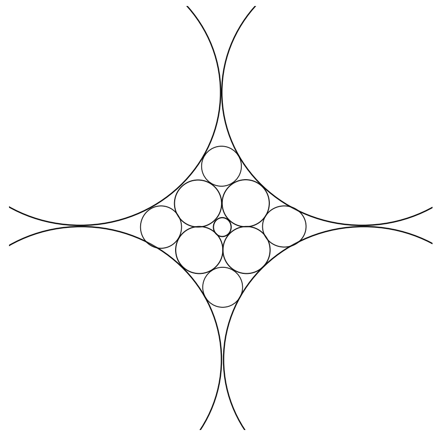


Figure 6: Central Quadrilateral Subdivision

Definition 2.1. (Central Quadrilateral Subdivision) Take any quadrilateral and insert a quadrilateral inside of it. At each vertex of the new quadrilateral attach an edge extending outward. At the other end of the new edge, create two edges branching off that connect on either side of a vertex of the original quadrilateral (see Figure 6).



(a) Subpacking of circle with 4 points of tangency



(b) After central quadrilateral subdivision on circle with 4 points of tangency

Figure 7: Central Quadrilateral Subdivision performed on a circle packing

Remark 2.2. Performing the central quadrilateral subdivision on a quadrilateral in the crushtacean of a polyhedral FAL results in altering the circle packing as described in Figure 7b.

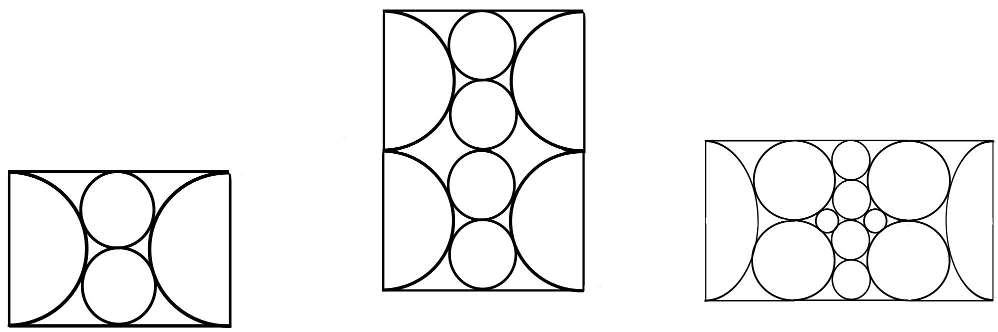
Theorem 2.3. *The central quadrilateral subdivision can be performed on the crushtacean of any FAL containing a quadrilateral.*

Proof. Let F be an FAL with associated crushtacean T and let C be the circle packing corresponding to P_+ of F. Then we know there exists some subpacking containing a circle with four points of tangency. Pick one such point of tangency, call it p, and send it to infinity. Then our subpacking will look like Figure 8a. We can then glue two cubeoctahedrons along a shaded edge to get Figure 8b. This introduces a problem though, as there are now four shaded regions that are quadrilaterals, two in the center, and two along the left and right vertical edge, which is not allowed for fundamental polyhedra. We remedy this by placing a circle in the two interstices in the middle, and by gluing half of a cubeoctahedra on either side of the packing as described in Figure 8c. When we pull p back from infinity, we get the subpacking as described in Figure 4 as desired.

□

Corollary 2.4. The central quadrilateral subdivision can be performed indefinitely.

Proof. In order to perform the central quadrilateral subdivision we must have a FAL that contains a quadrilateral. Since the central quadrilateral subdivision always introduces 5 new quadrilaterals, as long as no other procedure has been introduced, the central quadrilateral subdivision can always be performed. $\square$



(a) Subpacking with a point at infinity

(b) Gluing another copy of the subpacking along a shaded region

(c) Subpacking after gluing four circles to the quad-interstices

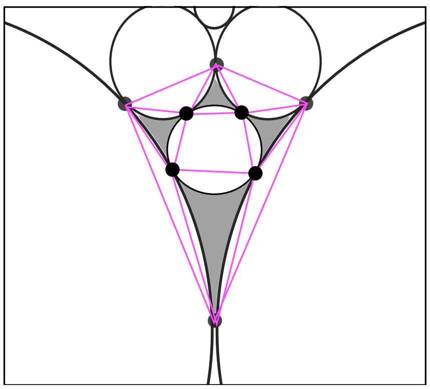
Figure 8: Central Quadrilateral Subdivision performed on an FAL

3 A Geometric View of Cuboctahedra

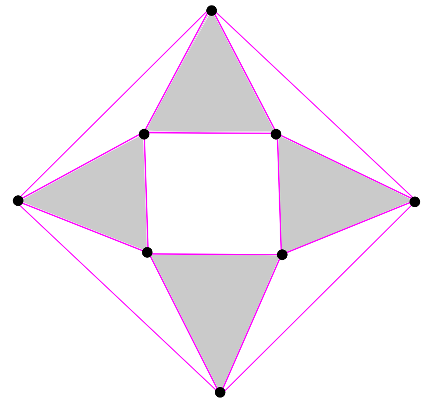
We now shift towards a geometric view of cuboctahedral FALs with the goal of understanding the way in which cuboctahedra are glued through Central Quadrilateral Subdivision.

Lemma 3.1. If two cuboctahedra, A and B, are glued along an unshaded face U, then the resulting shaded quadrilateral is a square.

Proof. Consider two cuboctahedra, A and B, glued along an unshaded quadrilateral U. By Lemma 1.12, there exists a cuboctahedron G made of half of A and half of B. Thus G is made of two regular ideal 4-Antiprisms, resulting in



(a) Addition of a 4-antiprism



(b) The 4-antiprism after rescaling

Figure 9

a regular ideal cuboctahedron. Thus the shaded quadrilateral faces of G are squares. $\square$

Lemma 3.2. A circle added to a shaded square interstice whose edges are those of distinct quadrilaterals produces a regular ideal 4-antiprism.

Proof. Consider a square interstice, where each edge is that of a distinct quadrilateral. Adding a circle that is mutually tangent to the edges of the existing quadrilateral interstice yields four new points of tangency, thus producing a new unshaded quadrilateral. Connecting consecutive points of tangency yields eight triangles, with 4 being shaded and 4 unshaded, resulting in a 4-Antiprism as seen in Figures 9a and 9b. The angles between shaded triangles and the unshaded quadrilateral are always $\frac{\pi}{2}$, and the angles between unshaded triangles and the shaded quadrilateral are $\frac{\pi}{2}$ by Lemma 3.1. Thus the resulting 4-Antiprism is a regular ideal 4-Antiprism. $\square$

Lemma 3.3. Central Quadrilateral Subdivision produces six new cuboctahedra.

Proof. We begin by considering the circle packing of P^+, C. Without loss of generality, consider the innermost circle of C, I. By reflecting C across the

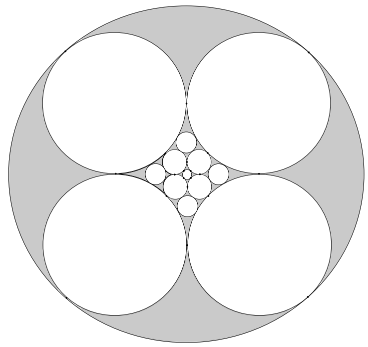


Figure 10: Circle packing after Central Quadrilateral Subdivision

plane bounded by I, we obtain a new cuboctahedron, N, that is glued to an unshaded quadrilateral face of P^+.

I is no longer present because this face is interior to the polyhedron $P^+ \cup N$. Accounting for the reflection in P^-, we have a current total of 2 additional cuboctahedra as seen in Figure 4b.

Now consider an interstice between P^+ and N in our circle packing. Note that this is a shaded square whose edges are those of distinct quadrilaterals. By Lemma 3.2, adding a circle to this interstice results in a regular ideal 4-antiprism, connecting to its reflection in P^- to form a cuboctahedron with the familiar shading. Repeating this process in each quadrilateral interstice yields four new cuboctahedra.

The circle packing obtained from these processes is that of the Central Quadrilateral Subdivision as seen in Figure 10, thus the Central Quadrilateral Subdivision results in six new cuboctahedra. $\square$

Note in the above proof that a cuboctahedron produced by the gluing of two 4-antiprisms, one in P^+ and one in P^-, is a cuboctahedron glued to a shaded quadrilateral face. This is the shaded face of the cuboctahedron noticed by the gluing of two existing cuboctahedra, as in Lemma 1.12. Adding a cuboctahedron to each shaded quadrilateral face will return the entire polyhedron to its proper shading.

4 A New Class of Cuboctahedral FALs

By our combinatorial and geometric analysis of cuboctahedral FALs, we've found a family of cuboctahedral FALs different from Worrell's 1st Family.

Theorem 4.1. *Performing the central quadrilateral subdivision on the crushtacean of a cuboctahedron with a quadrilateral unshaded face part of a 4-antiprism produces a cuboctahedral FAL.*

Proof. Let F be a cuboctahedral FAL where the associated circle packing C of P_+ of F contains a circle with 4 points of tangency. By Theorem 2.3, we know that if F contains a quadrilateral, we can perform the central quadrilateral subdivision on it. Further, by Lemma 3.4, we know this process adds exactly 6 new cuboctahedra to F. Then since F initially decomposed into cuboctahedra, F will still be cuboctahedral. □

Remark 4.2. Part of the central quadrilateral subdivision requires two cuboctahedra to be glued together along unshaded edges. This process is distinct of octahedral FALs, as Purcell rules out the possibility of a similar gluing pattern in her work [2].

We claim that the central quadrilateral subdivision produces an infinite family of cuboctahedral FALs that are distinct from the 1st Family. Before we prove this, we will prove a short lemma related to the crushtacean of any FAL belonging to the 1st Family.

Lemma 4.3. The crushtacean belonging to any FAL in the 1st Family must contain the vertex hexagon expansion subgraph.

Proof. Let F be an FAL belonging to the first family, that is not the simple cuboctahedral FAL. The only way to get an FAL belonging to the first family is by performing some number of vertex hexagon expansions on the crushtacean of the simple cubeoctahedral FAL. Then it must be the case that the last move performed to get F must have been a vertex hexagon expansion. This process replaces a vertex with the hexagon subgraph, meaning F contains the hexagon expansion subgraph. If F is the simple cuboctahedral FAL, then the whole graph is exactly the hexagon expansion with the 3 outer edges meeting at the same vertex. □

Theorem 4.4. *The central quadrilateral subdivision does not produce members of the first family.*

Proof. By Lemma 2.5, in order to be 1st Family, you must contain the hexagon expansion subfigure as part of your graph. The crushtacean of the simple cuboctahedron does contain the hexagon subfigure, however, by performing the central quadrilateral subdivision on any quadrilateral, it is now the case that every quadrilateral in the new crushtacean is adjacent to a hexagon. In order to contain the hexagon expansion subfigure, we must have 3 adjacent quadrilaterals, which is not possible in the current crushtacean. Further, while any additional iterations add 5 quadrilaterals to the crushtacean, none of the five are adjacent to each other. Additionally, the central quadrilateral subdivision must be performed on an existing quadrilateral in the crushtacean. Since no quadrilateral is adjacent to any existing quadrilateral, the new quadrilaterals added by the central quadrilateral subdivision will not be adjacent to any other quadrilaterals, for any number of iterations of the central quadrilateral subdivision. Therefore, it is not possible for the central quadrilateral subdivision to generate a member of the first family, as the resulting crushtacean does not contain the hexagon expansion subfigure. □

5 Further Questions

- Are the two moves outlined in this paper the only way to create cuboctahedral FALs?

  The cuboctahedral FALs belonging to the 1st Family classify all FALs that decompose into cuboctahedra such that no cuboctahedron is split between P_+ and P_-. We know the cuboctahedra formed by the central quadrilateral subdivision can be decomposed in such a way that cuboctahedra are split between P_+ and P_-, however, it's unclear if this is the only move capable of creating cuboctahedra that split across the reflection plane.

Additionally, preliminary results suggest that combining the vertex hexagon expansion with the central quadrilateral subdivision also creates cuboctahedral FALs.

- Is there a way to glue unshaded regions together for octahedra?

  In both Worrell and Purcell's papers, the move of gluing octahedral and cuboctahedral FALs to each other along unshaded regions is deemed to be impossible, as it affects the shading pattern, in the case of cuboctahedra, we end up with shaded quadrilaterals, which are not possible in $P_{+/-}$ by Purcell 2.2. In the cuboctahedral case, we were able to correct this by adding circles to each quad-interstice. Is there a corresponding move then that would allow the gluing of unshaded regions in the octahedral case? Preliminary results were unsuccessful in finding a corresponding move for cuboctahedra.

- What do all of the embedded totally geodesic surfaces look like in a cuboctahedral FAL that comes from Central Quadrilateral Subdivision?

  We saw in Lemma (?) that an embedded totally geodesic surface can be realized by the gluing of two regular ideal cuboctahedra. This helped us understand how and where the cuboctahedra were glued that came from the gluing of an unshaded circle to a shaded quadrilateral interstice. Thus, it would be interesting to know what the embedded totally geodesic surfaces looked like after central quadrilateral subdivision to gain new information.

- What happens if we belt-sum FALs from the two different families?

6 Acknowledgements

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