THE CATEGORY OF MODEL SPACES AND APPLICATIONS TO STRUCTURE GROUPS OF ALGEBRAIC CURVATURE TENSORS

LOGAN BORGHI

ABSTRACT. In order to study an Algebraic Curvature Tensor it is a known technique to instead study the structure of its Model Space. In this paper we develop a general theory for the category of Model Spaces.

1. Introduction: Basic Notions

To begin we define model spaces, which will be our primary object of study throughout this paper. Heuristically, our goal will be to better understand an algebraic curvature by looking at the structure of its model space.

Definition 1 (Model Space)**.** Let V be an $\mathbb{F}$-vector space and $R \in \mathcal{A}(V)$ an algebraic curvature tensor. We call the pair (V, R) a model space.

One can easily imagine by analogy to inner product spaces that the algebraic curvature tensor on a model space informs some geometry on its underlying vector space, much in the same way an inner product defines lengths and angles. Indeed this turns out to be the case and serves as a good intuition to keep in mind throughout.

Definition 2 (Kernel of ACT)**.** Let (V, R) be a model space. We define the subspace

$$\ker(R) := \{v \in V \ \mid \ R(v, \cdot, \cdot, \cdot) = 0\}$$

By a future result, any model space may be decomposed into the sum of the kernel and another subspace, thus the study of the structure of model spaces reduces to that of model spaces with trivial kernel. So from now on we will take all model spaces (V, R) to have $\ker(R) = \{0\}$.

Now we will make precise the notion of decomposition discussed above.

Definition 3 (Direct Sum of Model Spaces)**.** Let (V, R) and (W, S) be model spaces. We define the direct sum of model spaces $(V \oplus W, R \oplus S)$ by defining the algebraic curvature tensor

$$(R \oplus S)(v_i, w_i) := R(v_i) + S(w_i)$$

Proposition 1. One easily checks $R \oplus S \in \mathcal{A}(V \oplus W)$ so the direct sum of model spaces is a model space.

As when studying any mathematical object, one of the first questions to ask is what is the "correct" notion of sub-object. This question equally applies in the study of model spaces. A naive approach would be to define "sub-model spaces" as vector subspaces of the model space for which restricting the algebraic curvature tensor to defines a model space. This,

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however, is not the correct notion, as it turns out that *every* subspace would be a sub-model space, so that construction does not capture any structure of the algebraic curvature tensor.

It is not such a huge surprise though that "sub-model spaces" do not turn out to be the correct notion. In the category of rings for example, one never speaks of sub-rings but rather of ideals. We now define the right sub-object for a model space.

Definition 4 (Ideal)**.** Let (V, R) be a Model Space. We call a linear subspace $W \subseteq V$ an Ideal of V if there exists $\tilde{W} \subseteq V$ such that

$$(V, R) = (W, R|_W) \oplus (\tilde{W}, R|_{\tilde{W}})$$

Many interesting questions involving the ideal structure of model spaces quickly arise. For instance is the direct sum complement in the above expression unique provided it exists? Or more generally do model spaces admit a unique decomposition into indecomposable ideals?

Rather than attempting to answer these questions upright, we will take some time to develop some tools which we will use to attack these problems. First we need to define a Lie group called the structure group which is typically associated to a model space. We will be more concerned with the connected component of the identity of the structure group for reasons which will later become clear.

Definition 5 (Structure Group)**.** Let (V, R) be a model space. We define

$$G_R(V) := \left\{ A \in GL(V) \;\middle|\; A^*R = R \right\}$$

And denote the connected component of the identity by $G_R^0(V)$.

The structure group associated to a model space has been studied extensively. It is quite common to associate group invariants to objects which capture some algebraic data of a space. For example the fundamental group of a topological space which famously captures information about the holes of the space, or cohomology which captures some information about the torsion. The natural question then arises of what type of data is the structure group capturing about the model space it is built from. Perhaps the main success of the abstract categorical approach taken throughout this paper is that it gives a satisfying answer to that question.

Likely the most important question to ask regarding mathematical objects it what the right sense of "morphism" between those objects is.

Definition 6 (Morphism)**.** A linear map $\phi : (V, R) \longrightarrow (W, S)$ is a morphism if for every $A \in G_R^0(V)$ there exists $B \in G_S^0(W)$ such that the following diagram commutes.

$$\begin{tikzcd}
(V, R) \arrow[r, "\phi"] \arrow[d, "A"'] & (W, S) \arrow[d, "B"] \\
(V, A^*R) \arrow[r, "\phi"] & (W, B^*S)
\end{tikzcd}$$

Example 1.

(1) Any linear map $(V, R) \to (V, 0)$ is a morphism.
(2) The identity Id: $(V, R) \to (V, \lambda R)$ is a morphism.

Note that in both examples we may take $B = A$ to satisfy the definition of morphism (check this!) but for different reasons.

Proposition 2.
(1) If $\phi : (V, R) \to (W, S)$ and $\psi : (W, S) \to (Z, T)$ are morphisms, then the composition $\psi \circ \phi : (V, R) \to (Z, T)$ is a Morphism.
(2) The identity map $\mathrm{Id}_{(V,R)} : (V, R) \to (V, R)$ is a morphism.

Proof. Below we see for every $A \in G^0_R(V)$ there exists $C \in G^0_T(Z)$ commuting the diagram.

$$\begin{tikzcd}
(V,R) \arrow[r, "\phi"] \arrow[d, "A"'] & (W,S) \arrow[r, "\psi"] \arrow[d, "B"] & (Z,T) \arrow[d, "C"] \\
(V,A^*R) \arrow[r, "\phi"] & (W,B^*S) \arrow[r, "\psi"] & (Z,C^*T)
\end{tikzcd}$$

□

Corollary 1. Model spaces with morphisms as defined forms a category.

Now that we have built a category where we have sub-objects and morphisms, it seems sensible to aim to understand what the notion of equivalence we ought to mean in this context.

Definition 7 (Isomorphism)**.** We call a bijective morphism $\phi : (V, R) \to (W, S)$ with ϕ^{-1} a morphism an isomorphism of model spaces.

We remark that the condition ϕ^{-1} being a morphism is necessary, in the sense that there exists bijective morphisms whose inverses are not morphisms. Namely consider the identity morphism in example 1.Also note that this definition of isomorphism is different from the definition currently in use (that $\phi^*S = R$)

The following theorem is a nice characterization result which quantifies exactly what is meant by an isomorphism of model spaces in terms of their structure groups.

Theorem 1. A linear isomorphism $\phi : (V, R) \to (W, S)$ is an isomorphism of model spaces if and only if $G_R(V) = \phi^{-1}G_S(W)\phi$.

Proof. ($\Longrightarrow$) We sketch a proof.

$$\begin{aligned}
\phi \text{ isomorphism } &\Longrightarrow \forall A \in G^0_R(V) \quad \exists B \in G^0_S(W) \text{ such that } A = \phi^{-1}B\phi \\
&\Longrightarrow G^0_R(V) \subseteq \phi^{-1}G^0_S(W)\phi
\end{aligned}$$

Then by symmetry we get the reverse containment giving $G^0_R(V) = \phi^{-1}G^0_S(W)\phi$.
($\Longleftarrow$) Diagram chasing the definition of morphism, we pick $B = \phi^{-1}A\phi$. Checking $B \in G^0_S(W)$ completes the proof. □

2. Kernels and Ideals

Next we turn to examine the kernels of morphisms. For a morphism between model spaces we define the kernel to be the kernel of the underlying linear map. We now define another operation on model spaces, quotienting by the kernel of a morphism.

Definition 8 (Quotient by Kernels)**.** Let $\phi : (V, R) \to (W, S)$ be a morphism. We define $(V/\ker(\phi), \overline{R})$ by

$$\overline{R}(v_i + \ker(\phi)) = S(\phi(v_i))$$

Proposition 3.

(1) $\overline{R} \in \mathcal{A}(V/\ker(\phi))$ so that $(V/\ker(\phi), \overline{R})$ is a model space.

(2) The natural projection $\pi : (V, R) \to (V/\ker(\phi), \overline{R})$ by $v \mapsto v + \ker(\phi)$ is a morphism.

Proof. (1) First to check that $\overline{R}$ is well defined, we see that if we take different representatives $v_i + \ker(\phi) = v_i' + \ker(\phi)$, then the difference $v_i' - v_i \in \ker(\phi)$, so

$$\begin{aligned}
\overline{R}(v_i + \ker(\phi)) &:= S(\phi(v_i)) \\
&= S(\phi(v_i) + 0) \\
&= S(\phi(v_i + \phi(v_i' - v_i)) \\
&= S(\phi(v_i')) \\
&=: \overline{R}(v_i' + \ker(\phi))
\end{aligned}$$

thus $\overline{R}$ is well-defined. That $\overline{R} \in \mathcal{A}(V/\ker(\phi))$ follows from S being an ACT.

(2) To check π is a morphism, we pick an $A \in G^0_R(V)$ and find a $B \in G^0_{\overline{R}}(V/\ker(\phi))$ which commutes the following diagram.

$$\begin{tikzcd}
 & v \arrow[r, mapsto] & v + \ker(\phi) \\
v \arrow[d, mapsto] & (V, R) \arrow[r, "\pi"] \arrow[d, "A"'] & (V/\ker(\phi), \overline{R}) \arrow[d, "B"] \\
Av & (V, A^*R) \arrow[r, "\pi"'] & (V/\ker(\phi), \overline{R}) \\
 & Av \arrow[r, mapsto] & Av + \ker(\phi)
\end{tikzcd}$$

□

This requires us to find an automorphism of the quoitent space sending $v + \ker(\phi) \mapsto Av + \ker(\phi)$. To prove A induces a map $\overline{A}$ on the quotient space we must show that $\ker(\phi)$ is an invariantsubspace under A. But indeed it is, since ϕ is a morphism we see

$$\begin{aligned}
x \in \ker(\phi) &\implies \phi(Ax) = B\phi(x) = B0 = 0 \\
&\implies Ax \in \ker(\phi)
\end{aligned}$$

Thus $\ker(\phi)$ is an invariant subspace under the action of A so we may define $B = \overline{A} : V/\ker(\phi) \mapsto V/\ker(\phi)$ by $\overline{A}(v + \ker(\phi)) := Av + \ker(\phi)$, which the above computation demonstrates is well defined, and a routine computation checks indeed $\overline{A} \in G^0_{\overline{R}}(V/\ker(\phi))$, since $A \in G^0_R(V)$.

We will later greatly generalize this construction by quotienting by Ideals rather than simply kernels of morphisms, but it suffices for the following theorem.

Theorem 2 (First Isomorphism Theorem)**.** Let $\phi : (V, R) \to (W, S)$ be a morphism between model spaces. Then we have the following isomorphism of model spaces

$$\left(V/\ker(\phi), \overline{R}\right) \cong \left(\phi(V), S|_{\phi(V)}\right)$$

And the following diagram commutes

$$\begin{tikzcd}
(V, R) \arrow[rr, "\phi"] \arrow[d, two heads, "\pi"'] & & (W, S) \\
(V/\ker(\phi), \overline{R}) \arrow[urr, hook, "\overline{\phi}"'] & &
\end{tikzcd}$$

Proof. To exhibit the isomorphism we define

$$\Phi : (V/\ker(\phi), \overline{R}) \longrightarrow \phi(V), S|_{\phi(V)}) \text{ by } \Phi(v + \ker(\phi)) := \phi(v)$$

which we claim is a well defined isomorphism of model spaces. That this is a well defined vector space isomorphism is well known from linear algebra, thus it remains to check that Φ is in addition an isomorphism of model spaces.

We life the induced $\overline{A}$ on the coset space to A acting on (V, R) which we may do by having A fix $\ker(\phi)$ identically (which would then be an element of $G_R^0(V)$).

$$\begin{tikzcd}
& v + \ker(\phi) \arrow[r, mapsto] & \phi(v) & \\
v + \ker(\phi) \arrow[d, mapsto] & (V/\ker(\phi), \overline{R}) \arrow[r, "\Phi"] \arrow[d, "\overline{A}"'] & (\phi(V), S|_{\phi(V)}) \arrow[d, "B"] & \phi(v) \arrow[d, mapsto] \\
\overline{A}(v + \ker(\phi)) & (V/\ker(\phi), \overline{R}) \arrow[r, "\Phi"'] & (\phi(V), S|_{\phi(V)}) & B\phi(v) \\
& Av + \ker(\phi) \arrow[r, mapsto] & \phi(Av) &
\end{tikzcd}$$

Then the diagram commuting follows from the fact that ϕ is a morphism so there exists B such that $\phi \circ A = B \circ \phi$. □

Corollary 2 (Rank-Nullity)**.** Let $\phi : (V, R) \longrightarrow (W, S)$ be a morphism of model spaces. The following standard ismorphism result holds

$$(V, R) \cong (\ker(\phi), R|_{\ker(\phi)}) \oplus (\phi(V), S|_{\phi(V)})$$

Proof. □

From the last two results we get to connect the two seemingly very different notions of Ideals and Morphisms by the following theorem. The Kernel of a morphism of model spaces is an ideal, and every ideal may be exhibited as the kernel of a morphism of model spaces.

Theorem 3 (Kernels = Ideals)**.** Let $\phi : (V, R) \to (W, S)$ be a morphism of model spaces, then $\ker(\phi) \subseteq V$ is an ideal, and conversely every ideal may be exhibited as the kernel of some morphism between model spaces.

Proof. The foreword direction follows immediately from Rank-Nullity. For the reverse definition we consider the natural projection

$$\pi : (V, R) \oplus (W, S) \to (V, R) \quad (v, w) \mapsto v$$

We claim this natural projection is a morphism (proposition 6) of model spaces, the kernel of which is clearly the ideal (W, S). ☐

3. R-Perp Spaces

A surprisingly useful tool to detect and study the ideal structure of a model space is the notion of R-Perp spaces. The data of the ACT turns out to mirror closely that of an inner product and many results of inner products carry over to this setting.

Definition 9 (R-Perp Space)**.** Let (V, R) be a Model Space and $W \subseteq V$ a linear subspace. We define

$$W_R^\perp := \{v \in V | R(v, -, -, W) = 0\}$$

Proposition 4.

(1) $W_R^\perp$ is a linear subspace of V.
(2) $\{0\}_R^\perp = V = \ker(R)_R^\perp$
(3) $W_R^\perp \cap W \subseteq \ker(R)$
(4) $W \subseteq W' \implies W_R'^\perp \subseteq W_R^\perp$

Proof. It is straightforward from the definition of $R-$perp spaces to verify (1)-(4). ☐

Theorem 4 (Uniqueness of Coideals)**.** Let (V, R) be a Model Space and W an Ideal of V. Then W has a unique coideal $\tilde{W}$ such that $(V, R) \cong (W, R|_W) \oplus (\tilde{W}, R|_{\tilde{W}})$. Furthermore, $\tilde{W} = W_R^\perp$.

Proof. We sketch the proof. Suppose we have a direct sum decomposition $(V, R) = (W_1, R_1) \oplus (W_2, R_2)$ then we claim $W_2 \subseteq (W_1)_R^\perp$ and since $((W_1)_R^\perp, R_\perp) \cap (W_2, R_2) \subseteq \ker(R) = 0$ we get that $(W_1)_R^\perp$ is another coideal of W_1 so they have the same dimension. But if a subspace is contained in another of the same dimension then they must be equal. ☐

Theorem 5 (Isomorphisms Respect R-Perp Spaces)**.** Let $\phi : (V, R) \xrightarrow{\sim} (W, S)$ be an isomorphism of Model Spaces, and $U \subseteq V$. Then

$$\phi(U_R^\perp) = \phi(U)_S^\perp$$

Proof. It is easy to show mutual inclusion. ☐

Theorem 6 (R-Perp Space is Ideal)**.** Let (V, R) be a Model Space and $W \subseteq V$ a linear subspace. Then $W_R^\perp$ is an Ideal, with

$$(V, R) = (W_R^\perp, R|_{W_R^\perp}) \oplus ((W_R^\perp)_R^\perp, R|_{(W_R^\perp)_R^\perp})$$

and $(W_R^\perp)_R^\perp$ is the smallest ideal of V containing W. In particular,

$$W \text{ is an Ideal} \iff (W_R^\perp)_R^\perp = W$$

Proof. ☐

Having briefly explored the concept of ideals of model spaces, we are prepared to introduce a key type of morphisms, projections. We will later see that projections form the basic structure of morphisms.

Proposition 5. Let $\pi : (W, R|_W) \oplus (W', R|_{W'}) \to (W, R|_W)$ be the canonical projection onto the ideal W. Then π is a morphism of model spaces.

Definition 10 (Prime Ideal)**.** We call an ideal W of a model space (V, R) with trivial kernel prime if W contains no nontrivial ideals.

Conjecture 1. A model space (V, R) decomposes uniquely (upto reordering) into

$$(V, R) = \bigoplus_i (W_i, R|_{W_i})$$

with W_i prime ideals.

In all known cases this result holds, and we note that such a prime decomposition always exists by dimension arguments. It only remains to prove that such decompositions are unique.

The next theorem suggests the existence of a functor from the category of Model Spaces to Lie Groups.

Theorem 7. The reduced structure group respects ideal decompositions of a model space. That is,

$$G^0_{R\oplus S}(V \oplus W) \cong G^0_R(V) \oplus G^0_S(W)$$

Proof. Check the map $(A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is an isomorphism of lie groups. Clearly it is an injective, smooth, group homomorphism. The difficulty lies in showing this map is surjective. □

Definition 11 (Structure Group Pushforeward)**.** Let $\phi : (V, R) \longrightarrow (W, S)$ be a morphism. We define the pushforeward $\phi_* : G^0_R(V) \longrightarrow G^0_S(W)$ between the corresponding reduced structure groups by

$$\phi_*(A) = (\phi|_{\ker(\phi)^\perp_R} \circ A|_{\ker(\phi)^\perp_R} \circ \phi|^{-1}_{\ker(\phi)^\perp_R}) \oplus \mathbb{I}_{\dim(W)-\dim\phi(V)}$$

The definition comes from diagram chasing the following diagram

$$\begin{tikzcd}
(V, R) \arrow[r, "A"] & (V, A^*R) \\
(\ker(\phi)^\perp_R, R_\perp) \arrow[u, "i"] \arrow[r, "\overline{A}"'] \arrow[d, "\phi|_{\ker(\phi)^\perp_R}"'] & (\ker(\phi)^\perp_R, \overline{A}^* R_\perp) \arrow[u, "i"'] \arrow[d, "\phi|_{\ker(\phi)^\perp_R}"] \\
(\phi(V), S|_{\phi(V)}) \arrow[r, "\widetilde{A}"'] & (\phi(V), \widetilde{A}^* S|_{\phi(V)})
\end{tikzcd}$$

We define $\phi_*(A)$ to be $\widetilde{A} \oplus \mathbb{I}$.

Theorem 8 (Pushforeward is a Functor)**.**

Proof. One checks ϕ_* is a well-defined homomorphism of Lie groups. □

Proposition 6.

(1) $(\mathrm{Id}_{(V,R)})_* = \mathrm{Id}_{G_R(V)}$
(2) If $\phi : (V, R) \to (W, S)$ and $\psi : (W, S) \to (Z, T)$ are morphisms, then $(\phi \circ \psi)_* = \phi_* \circ \psi_*$.

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