# $b$-prime Fully Augmented Links and Complete Augmenation

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**Abstract**

Fully augmented links are powerful tools to study hyperbolic links. In particular, b-prime FALs, or fully augmented links which do not admit belted sum decompositions, are building blocks for FALs. We consider complete augmentation, a method to generate FALs. First, we show that an FAL is b-prime if and only if it is the complete augmentation of a prime, nonsplittable link diagram. Second, we show that every b-prime FAL is the complete augmentation of some prime, non-splittable link diagram.

## 1 Introduction

This paper focuses on the study of fully augmented links (FALs). Consider a *twist region*, or a region in a link diagram composed of either a maximal string of bigons or a single vertex adjacent to no bigons. We augment a link by placing a crossing circle around a twist region. Correspondingly, a link is *fully augmented* if all twist regions are augmented. FALs historically arose from the study of manifolds, as the majority of FALs have hyperbolic complements. Since performing a Dehn twist leaves the complement homeomorphic to the complement of the original FAL, we will assume that each crossing circle bounds zero or one crossings. In an appendix to [**?**], I. Agol and D. Thurston used decomposition of FAL complements into ideal polyhedra to study the volume of the link complements. Adams was the first to generally study the geometry of FALs, in [**?**].

FALs have varied uses, as can be seen in [**?**], [**?**], and [**?**]. In general, FAL complements have hyperbolic structures which are relatively easy to work with. Just as we can categorize links as prime and composite, we can characterize FALs as belted sum prime and composite. $b$-prime FALs form the building blocks for FALs insofar as their combinations respect the interaction of certain invariants one would expect. Most notably, volume adds with belted sums [**?**].

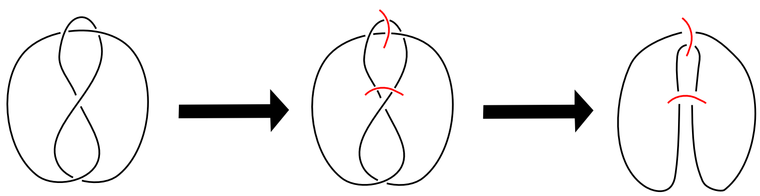

Figure 1: Full augmentation of $4_1$ into the Borromean rings

Since understanding $b$-prime FALs helps us to understand FALs overall, we would like to have a method to generate $b$-prime FALs. To this extent, we introduce *complete augmentation*, first developed by Mork in [?]. Whereas full augmentation places a crossing circle around each twist region, complete augmentation places a crossing circle around each crossing. This procedure always can be performed on prime links (Lemma **??**), and for a prime, non-splittable link diagram with at least three crossings, complete augmentation produces a hyperbolic FAL (Lemma **??**). In particular, an FAL is $b$-prime if and only if it is the complete augmentation of a prime, non-splittable link diagram with at least three crossings (Theorem **??**). With all of this, it comes as no surprise that all $b$-prime FALs can be generated as the complete augmentation of a prime, non-splittable link (Theorem **??**).

In Section **??**, we define belted sums and give some properties of belted sums and $b$-prime FALs. Then in Section **??**, we prove our main results about complete augmentations. These two sections are a more thorough exposition and an extension of the work in [?]. One application of our results is a bound on the number of FALs which can be generated from a given link diagram. We explain this in Section **??**, and in Section **??**, we provide some related conjectures and directions for future work.

Note that for the duration of the paper, we will assume that all link diagrams are reduced and non-splittable, and that all disks are open. Also, we will denote the $n$ crossing circles of an FAL $L$ by $C_1, \ldots, C_n$.

### 1.1 Acknowledgements

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## 2 Belted Sums and $b$-prime Links

This section defines belted sums and $b$-prime links and establishes some basic properties. The primary method of detecting $b$-prime FALs which we will use depends on finding composing tangles. To define these, we will introduce the notion of a link universe, which we use extensively in both this section and the remainder of the paper.

**Definition 1** (Belted Sum Decomposition ([?], Section 5))**.** We give a diagrammatic definition for a belted sum decomposition via the link diagrams in Figure **??**. We define $L$ to be the belted sum of $L_1$ and $L_2$, notated as $L_1 \#_b L_2$.

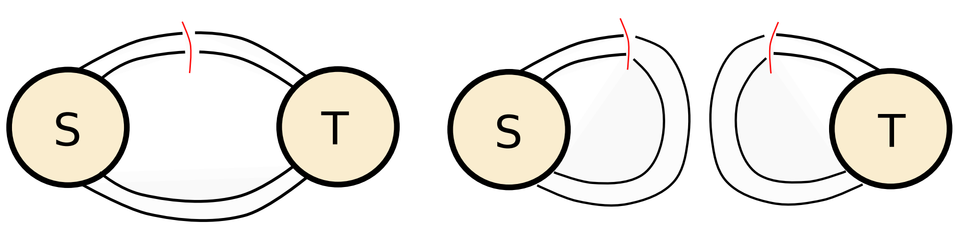



Figure 2: The canonical example of a belted sum decomposition

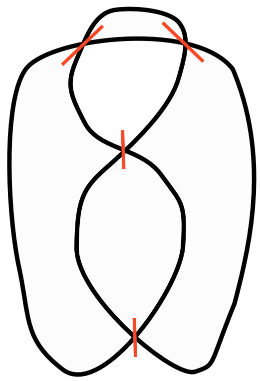


Figure 3: An example of a $b$-prime FAL

**Definition 2.** A **$b$-prime FAL** is a hyperbolic FAL with no belted sum decomposition.

One of the primary motivations for studying $b$-prime FALs comes from the following result, that volumes of link complements sum over belted sums.

**Theorem 1** (Adams, [**?**, Corollary 5.2])**.** *Suppose that $L_1$ and $L_2$ are links with hyperbolic complements, and let $v(L)$ denote the volume of $S^3 \setminus L$. Then $v(L_1 \#_b L_2) = v(L_1) + v(L_2)$.*

Hence, in order to understand the volumes of hyperbolic link complements, we can study $b$-prime FALs. To this end, we note that a half twist on the portion of a link bound by a crossing circle does not change the volume of the link's complement ([**?**], Corollary 5.1). Thus we can more or less equally consider flat FALs, i.e. FALs where all crossing circles are flat, and non-flat FALs. With $b$-prime FALs defined, we move towards defining composing tangles.

**Definition 3** (Link universe)**.** For link $L$, a **link universe**, denoted $\pi(L)$, is a projection of $L$ onto $S^2$ such that for $x \in \pi(L)$, $\pi^{-1}(x)$ has at most two points.

When producing **an FAL universe**, $\pi(L')$, we project crossing circles such that they are arcs in the plane. If a crossing circle bounds a twist, its projection must intersect that twist at the crossing. We refer to $\pi(C_i)$ as a *crossing arc* and to the components of $\pi(L' \setminus \{C_1, \dots, C_n\})$ as *link segments*.

**Definition 4** (Composing tangle)**.** An **$m$-tangle** of $\pi(L)$ is the portion of $\pi(L)$ bounded by an open disk $D$ such that

1. $\pi(L) \cap D$ is connected.
2. $\pi(L) \cap \partial D$ and $\pi^{-1}(\pi(L) \cap \partial D)$ have $m$ points.

In a link universe for a non-FAL link, a **composing tangle** is a 4-tangle. In an FAL universe $\pi(L')$, a **composing tangle** is a 4-tangle bounded by an open disk $D$ such that

1. $\pi(L') \cap \partial D$ contains two points and a crossing arc.

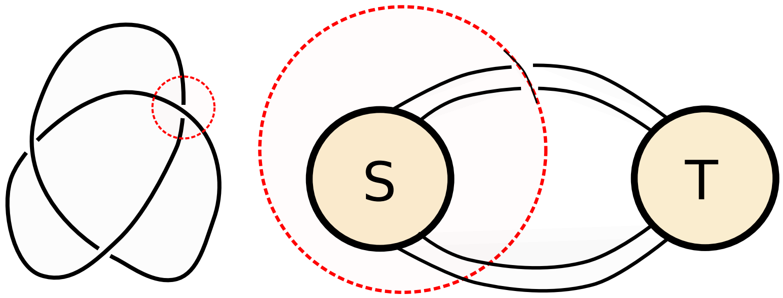



Figure 4: A composing tangle in a non-FAL link universe (left) and an FAL universe (right)

2. $\pi^{-1}(\pi(L' \setminus \{C_1, \ldots, C_n\}) \cap \partial D)$ has four points.

We say tangles are *trivial* in a link universe for a non-FAL link if they contain all or no crossings. In an FAL universe, tangles are *trivial* if they contain all or no crossing arcs. Examples of composing tangles in a link universe of a non-FAL link and in an FAL universe are shown in Figure **??**.

The notion of a flype parallels composing tangles, so we introduce that here.

**Definition 5** (Flypes, flype tangles, and flype orbits)**.** A **flype** is a 180° rotation of a composing tangle, such that a crossing moves across that tangle. We say that crossing induces the flype. Of the composing tangles which admit flypes, we call minimal ones **flype tangles**. The collection of flype tangles induced by a crossing is a **flype orbit**. A flype orbit is **trivial** if it contains only one flype tangle.

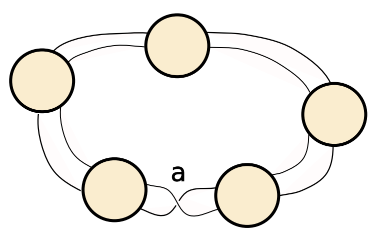



Figure 5: A flype orbit induced by crossing $a$

**Remark 1.** A link diagram $D(L)$ induces a non-trivial flype orbit if and only if it contains a non-trivial composing tangle.

**Theorem 2** (Morgan et. al., [**?**, Theorem 5])**.** *Let $D$ be a reduced, non-split, non-torus, prime, alternating link diagram with more than two twist regions, and let $A$ be a full augmentation of $D$. The fully augmented link $A$ is ps-prime if and only it induces only trivial flype orbits.*

**Remark 2.** As the authors in [**?**] note, the conditions on $D$ ensure that $A$ is hyperbolic.

We want composing tangles to effectively detect $b$-prime FALs. The following result tells they in fact do.

**Proposition 3.** *A hyperbolic FAL $L'$ is $b$-prime if and only if $\pi(L')$ has no non-trivial composing tangles.*

*Proof.*
($\Rightarrow$): Suppose $L'$ is a $b$-prime FAL, and suppose for the sake of contradiction that the composing tangle $T$ of $\pi(L')$ is non-trivial. Let $D$ be the disk bounding $T$. We can view $T$ in $L'$ as bounded by a sphere. Cutting along the boundary of that sphere gives a belted sum decomposition of $L'$, a contradiction. Hence, $T$ must be trivial.
($\Leftarrow$): Suppose that $L'$ is not $b$-prime. By Theorem **??** and Remark **??**, $\pi(L')$ admits a belted sum decomposition, and there exists a disk $D$ bounding non-trivial composing tangle $T$ in $\pi(L')$. □

## 3 Complete Augmentation

In this section, we define the notion of complete augmentation. First, we show that we can take the complete augmentation of any prime link diagram. Then, we work towards showing that any $b$-prime FAL can be generated by complete augmentation.

**Definition 6.** A **complete augmentation** $L'$ of a link diagram $D(L)$ is produced by adding a crossing circle around each crossing such that $\pi(L')$ contains only trivial composing tangles.

In order to show that any prime link diagram has a complete augmentation, we need a lemma about non-trivial flype orbits.

**Lemma 4** (Calvo, [**?**, Lemma 4])**.** *In a prime link universe, each crossing generates at most one non-trivial flype orbit.*

**Lemma 5.** *Every prime link diagram has a complete augmentation.*

*Proof.* By Proposition **??**, it suffices to show that for each crossing in a prime link, there exists a way to augment the crossing without introducing a non-trivial composing tangle. Suppose that for crossing $\alpha$, both ways of augmenting $\alpha$ produce a non-trivial composing tangle. By Remark **??**, $\alpha$ induces two non-trivial flype orbits. This contradicts Lemma **??**, so one way of augmenting $\alpha$ must avoid creating non-trivial composing tangles. □

Since $b$-prime FALs are hyperbolic, it would be good to know that complete augmentation produces hyperbolic FALs. The next result provides a criteria for detecting hyperbolic FALs.

**Theorem 6** (Purcell, [**?**, Theorem 2.5])**.** *An FAL $L'$ is hyperbolic if and only if the associated knot or link diagram $D(L)$ is non-splittable, prime, twist-reduced, with at least two twist regions.*

**Lemma 7.** *Any complete augmentation of a prime, non-splittable link diagram with at least three crossings is a hyperbolic FAL.*

**Remark 3.** There is only one prime link with two crossing circles, and only one complete augmentation is a hyperbolic FAL. The possible complete augmentations are shown in Figure **??**.

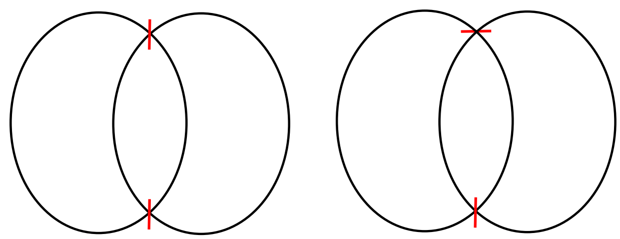


Figure 6: The left link admits an essential torus and so is not hyperbolic. The right link is hyperbolic.

*Proof.* Let $D(L)$ be a prime, non-splittable link diagram with $L'$ as its complete augmentation. Note that $L'$ exists by Lemma **??**. We perform Dehn fillings on each crossing circle such that crossing circles bound three crossings in the resulting diagram. We call the resulting link diagram $D(L^*)$. In $D(L^*)$, no two twist regions can combine, as $\pi(L')$ has no composing tangles. Hence, the full augmentation of $D(L^*)$ is $L'$. By Theorem **??**, it suffices to show that $D(L^*)$ is prime and twist-reduced, with at least two twist regions.

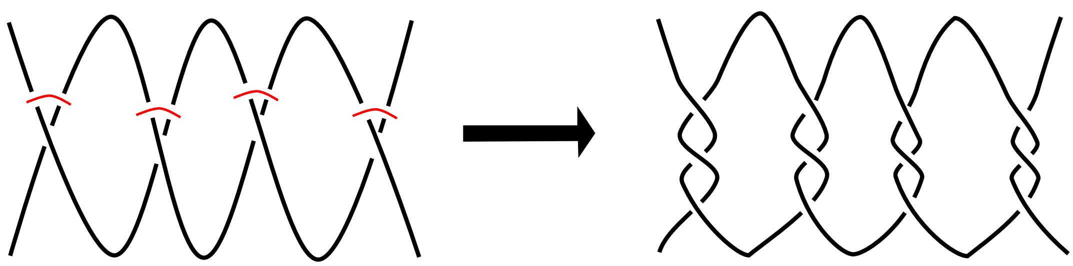
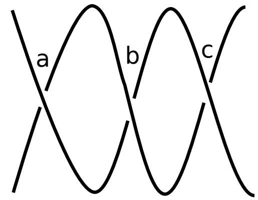


Figure 7: On the left is a twist region of $L'$. On the right is the corresponding twist region of $L^*$.

First we will show that $D(L^*)$ is prime. To do so, we first show that $D(L')$ is prime. Suppose for the sake of contradiction that $D(L')$ is not prime. Then $D(L')$ contains a non-trivial 2-tangle. Since complete augmentation preserves 2-tangles, $D(L)$ must contain a non-trivial 2-tangle as well, a contradiction. Thus $D(L')$ is prime. Now suppose for the sake of contradiction that $D(L^*)$ is not prime. Then $D(L^*)$ contains a non-trivial 2-tangle. Since full augmentation preserves 2-tangles, $D(L')$ must contain a non-trivial 2-tangle, a contradiction. Thus $D(L^*)$ is prime.

Now consider the twist regions of $D(L^*)$ and suppose for the sake of contradiction that $D(L^*)$ has exactly one twist region. Since $L'$ is a full augmentation of $D(L^*)$, $D(L')$ will have only one crossing circle. However, we assume that $L'$ has at least three crossing circles, a contradiction. Finally, suppose for the sake of contradiction that $D(L^*)$ is not twist reduced. We will show that for each twist region, there are no flype tangles which contain a proper subset of the crossings outside of the twist region. Note that in $L^*$, each twist region has precisely three crossings. Label those $a, b$, and $c$.




Since $b$ induces trivial flype orbits, we can consider the flype orbits of $a$ without loss of generality. First, $b$ and $c$ are contained in distinct non-trivial flype tangles in the flype orbit induced by $a$. Suppose that there exists a third flype tangle, which contains a proper subset of the crossings outside of the twist region. By Remark **??**, there is a non-trivial composing tangle in $D(L^*)$. But this composing tangle will persist in $D(L')$ after complete augmentation, which contradicts the construction of $L'$ as the complete augmentation of a link. Hence, $D(L^*)$ is twist reduced. □

**Theorem 8.** *A complete augmentation $L'$ of link diagram $D(L)$ with at least three crossings is b-prime if and only if $D(L)$ is prime and non-splittable.*

*Proof.*
($\Rightarrow$): Suppose that $L'$ is $b$-prime, and suppose by way of contradiction that $\pi(L)$ is not both prime and non-splittable. If $D(L)$ is splittable, then $D(L')$ is too. So suppose $D(L)$ is not prime, i.e. $D(L)$ has a non-trivial 2-tangle. But complete augmentation preserves 2-tangles, since for a disk $D$ bounding an $m$-tangle in a non-FAL link universe, $\partial D$ cannot contain a crossing. Hence $D(L')$ has a non-trivial 2-tangle as well. A non-trivial 2-tangle implies the existence of an essential annulus in the complement of the link, which contradicts Thurston's characterization of a hyperbolic link. This is a contradiction to the definition of a $b$-prime FAL, so $D(L)$ is prime as well.
($\Leftarrow$): Suppose $L$ is prime and non-splittable, with at least three crossings, and suppose by way of contradiction that $L'$ is not $b$-prime. By Proposition **??**, $L'$ contains a non-trivial composing tangle. However, this contradicts that $L'$ is a complete augmentation of $L$. □

Note that link diagrams need not be twist reduced. Moreover, we conjecture that the complete augmentation of a diagram and a flype of that diagram will be different FALs. Now that we know we can completely augment prime links and produce $b$-prime links. The natural question to ask is whether all $b$-prime links can be produced in this way. The following theorem states that they can.

**Theorem 9.** *Every b-prime FAL can be generated as a complete augmentation of the diagram of a prime, non-splittable link up to Dehn twists and addition of half twists.*

**Remark 4.** Adams ([**?**, Corollary 5.1]) shows that volume of link complements is invariant up to half twists. Hence, to understand $b$-prime FALs, it suffices to study them up to half twists. Note also that by Morgan et. al ([**?**, Theorem 4]), $b$-primality is invariant under the addition of half twists.

*Proof.* Consider a $b$-prime FAL $L'$. Since we can perform Dehn twists to, and add half twists on, the FAL, we can suppose that all crossing circles in the FAL bound a crossing. We want to show the existence of a prime, non-splittable link diagram $D(L)$ of which $L'$ is a complete augmentation. We will prove three claims.

Claim 1: $D(L' \setminus \{C_1, \ldots, C_n\})$ is reduced.
Suppose for the sake of contradiction that $D(L' \setminus \{C_1, \ldots, C_n\})$ is not reduced (Adams, [**?**, pp. 68]). Then there exists an easily removable crossing, which can be removed with a Reidemeister 1 move (Case 1) or by rotating a composing tangle to remove a central crossing (Case 2).

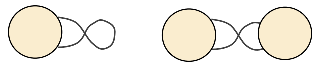


Figure 8: On the left is Case 1; on the right is Case 2.

In both cases, $L'$ is not hyperbolic, a contradiction to Lemma **??**. Hence, the diagram is reduced. Note that this matches our restriction to considering only reduced link diagrams.

Claim 2: $D(L)$ is non-splittable, where $D(L)$ is a link diagram with $L'$ as its complete augmentation.
Suppose $D(L)$ is splittable. Then we can embed an essential 2-sphere in the complement of $L'$, and $L'$ is not hyperbolic, a contradiction.

Claim 3: $D(L)$ is prime.
Suppose $D(L)$ is composite. Then there exists a non-trivial 2-tangle in $D(L)$, and hence a non-trivial 2-tangle in $D(L')$. Thus $L'$ is not composite, and $L'$ is not hyperbolic, a contradiction. □

## 4 Generating FALs from Prime, Non-splittable Link Diagrams

We want to understand how many FALs can be generated from a given prime, non-splittable link diagram with at least three crossings up to addition or removal of half twists. In general, this is a fairly difficult problem, so we'll consider a bound on the number of flat FALs which can be generated from such a link diagram. We define protected crossings to build an approach for this bound.

**Lemma 10.** *For a twist region $T$ with at least two crossings in prime link $L$ with at least three crossings, $T$ has a unique complete augmentation.*

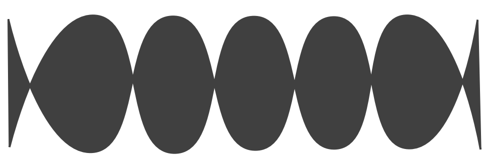


Figure 9: The region $R$

*Proof.* Take the region $R$ bounded by $\pi(T)$ in $\pi(L)$. We claim that the projection of the unique complete augmentation of $T$ has crossing arcs contained in $R$. Suppose there is a crossing arc $C$ in a complete augmentation of $T$ which is not contained in $R$. Without loss of generality, suppose that $C$ is not the last crossing arc in the tangle. Draw a disk $D$ in $\pi(L)$ such that $\partial D$ contains

$C$ and two points in knot segments between the crossing circle after $C$ and the crossing circle two after $C$. Note that $D$ bounds a non-trivial composing tangle, which contradicts the possibility of including $C$ in a complete augmentation of $T$. $\square$

**Definition 7.** In a link diagram, a crossing is **protected** if in the set of complete augmentations of the link, the crossing has two possible augmentations.

**Remark 5.** All protected crossings are in twist regions with only one crossing, by Lemma **??**

For an FAL with $n$ crossings, we suppose that the first $m$ of them are protected. We also suppose that each crossing for $k \in \{m+1, \ldots, n\}$ induces a flype orbit with $i_k$ flype tangles and $j_k$ crossings outside of flype tangles. Then we can show that.

**Proposition 11.** *Consider a twist-reduced prime, non-splittable link diagram with $n \geq 3$ crossings. The number of flat FALs which can be generated by complete augmentation of the diagram and ones flype equivalent to it, when complete augmentation is considered up to half twists, is $2^m \prod_{k \geq m+1}^{n} \binom{i_k+j_k-1}{j_k}$.*

*Proof.* First, note that each protected crossing contributes a factor of 2 to the number of flat FALs which can be generated by complete augmentation of the diagram, as irrespective of the rest of the link, there are two ways to augment a protected crossing. Each non-protected crossing must induce a flype orbit (otherwise it would be a protected crossing), so we can consider the number of ways to distribute crossings outside the flype tangles among the flype tangles. First note that flypes do not introduce or destroy protected crossings, as flypes preserve composing tangles. Since we want to bound the number of FALs produced, we allow overcounting, so we can assume that after each flype, there is a different flat FAL produced. Thus, the problem reduces to considering distributions of $j_k$ distinct balls over $i_k$ identical boxes, of which there are $\binom{i_k+j_k-1}{j_k}$ many by well-known results in enumerative combinatorics. Note that if a crossing induces a trivial flype orbit, this is analogous to evaluating $\binom{j_k}{j_k}$, so trivial flype orbits have no impact on the number of flat FALs we generate. $\square$

**Remark 6.** Without considering symmetries, the $2^m$ term in the bound is sharp! Consider the Borromean rings, L6a4 in Thistlewaite's Link Table. All crossings in the link are protected, so without considering symmetries, there are exactly $2^6$ complete augmentations of the Borromean rings resulting in flat FALs. However, considering symmetries, some of the resulting complete augmentations may be equivalent.

# 5 Further Work

We would like to improve the bound on flat FALs generated from a twist-reduced link diagram with $n$ crossings. Here are some potential approaches.

1. Consider symmetries of the link diagram before augmentation.
2. Find a sufficient characterization for protected crossings.
3. Alternatively, show that the bound without considering symmetries is sharp.

Separately, another direction of research would be building connections between the Tutte-Whitney polynomial of a crushtacean (see [**?**]) and the Jones polynomial of a corresponding $b$-prime FAL. This could be expanded to consider the effects of complete augmentation on the Jones polynomial, or links with certain conditions on their Jones polynomials which have nice properties in their Jones polynomial or the Tutte-Whiteny polynomial of their crushtacean after complete augmentation. To our knowledge, this second direction has not been explored.

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