The Denseness of Canonical Algebraic Curvature Tensors and a Revision to the Signature Conjecture

Kieran Favazza

Abstract

The structure of the space of algebraic curvature tensors over a vector space is of great interest and captures information about the behavior of the Riemann curvature tensor on a manifold. Studying this structure in itself and underlying sub-structures can additionally help in the determination of invariants of algebraic curvature tensors. These are of importance in understanding how algebraic curvature tensors can be distinguished from one another; we investigate invariants of algebraic curvature tensors to further our understanding of this topic in connection with the structure of the space of algebraic curvature tensors.

1 Introduction

It is the principle effort of differential geometry to distinguish and classify manifolds and analyze their properties. Among these properties, curvature is perhaps the deepest and most exciting. Given any Riemannian or pseudo-Riemannian manifold M, one can use the Levi-Civita connection to build the *Riemann curvature tensor* on M: a function acting on the vector fields of M that measures the curvature of the manifold. Given a point $p \in M$, the Riemann curvature tensor determines a multilinear function $R_p : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$, where T_pM is the tangent space of M at p, that satisfies certain algebraic properties. If we act without regard to any particular manifold and just consider arbitrary vector spaces, we can consider tensors acting on these vector spaces that satisfy the properties of the Riemann curvature tensor at a point. These algebraic portraits of curvature are interesting in their own right and give insight into the behavior of the Riemann curvature tensor on a manifold.

1.1 Algebraic Curvature Tensors

Definition 1. Throughout, let V denote a real vector space of finite dimension n. An *algebraic curvature tensor* on V is a multilinear function $R : V^4 \to \mathbb{R}$ such that for all $x, y, z, w \in V$,

(i) $R(x,y,z,w) = -R(y,x,z,w)$,

(ii) $R(x,y,z,w) = R(z,w,x,y)$,

(iii) $R(x,y,z,w) + R(z,x,y,w) + R(y,z,x,w) = 0$.

This last condition is called the *Bianchi Identity*. The set of algebraic curvature tensors over a vector space V is itself a vector space, denoted $\mathcal{A}(V)$. Further, note that because algebraic curvature tensors are multilinear, they can be specified by their values on some basis $\{e_i\}$ of V. We denote $R(e_i, e_j, e_k, e_l) = R_{ijkl}$.

We wish to study the behavior and structure of algebraic curvature tensors. In doing so, it is helpful to be aware of canonical constructions of algebraic curvature tensors. Particularly, we consider the tensors constructed from symmetric bilinear forms on the vector space V. Working with these forms specifically lends us many tools from linear algebra that can simplify the solutions and statements of questions.

Definition 2. A *symmetric bilinear form* φ on a vector space V is a mapping $\varphi : V \times V \to \mathbb{R}$ such that for all $x, y, z \in V$, $\varphi(x,y) = \varphi(y,x)$ and $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x,z) + \beta\varphi(y,z)$ for all $\alpha, \beta \in \mathbb{R}$. We denote the set of all symmetric bilinear forms on V as $S^2(V)$. Throughout, if we are given a basis $\{e_i\}$ of V, we identify the symmetric bilinear form φ with its matrix representation $[a_{ij}] = \varphi(e_i, e_j)$.

According to the Spectral Theorem, any symmetric bilinear form can be diagonalized on some basis, and Sylvester's Law of Inertia [8] says that the number of negative entries p, the number of positive entries q, and the number of zeroes s along the diagonal in any diagonalization are all unique. This allows the following definition.

Definition 3. For any $\varphi \in S^2(V)$, the *signature* of φ is the ordered triple (p, q, s).

We now define the canonical algebraic curvature tensors of symmetric build.

Definition 4. Given $\varphi \in S^2(V)$, the *canonical algebraic curvature tensor* R_φ is given by

$$R_\varphi(x,y,z,w) = \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w).$$

Note that for any $c \in \mathbb{R}$, if $c \geq 0$, then $cR_\varphi = R_{\sqrt{c}\varphi}$, and if $c < 0$, then $cR = -R_{\sqrt{|c|}\varphi}$.

Canonical algebraic curvature tensors, being built from the more digestible symmetric bilinear forms, are easier to understand in general than arbitrary algebraic curvature tensors. In [3], it was shown that

$$\mathcal{A}(V) = \operatorname{span}\{R_\varphi : \varphi \in S^2(V)\},$$

so it is in our interest to study how general algebraic curvature tensors can be expressed as linear combinations of canonical algebraic curvature tensors.

In [4], it was shown that $\dim(\mathcal{A}(V)) = \frac{n^2(n^2-1)}{12}$, so we can surely express each $R \in \mathcal{A}(V)$ as the sum of at most this many canonical algebraic curvature tensors since any spanning set of a vector space contains a basis, but could we use fewer? Studying this relationship illuminates the structure of canonical algebraic curvature tensors, and it additionally points towards the construction of invariants that can distinguish between algebraic curvature tensors. This motivates the following definition.

Definition 5. Let $R \in \mathcal{A}(V)$. We define the quantities

$$\nu(R) = \min\{k : R = \sum_{i=1}^{k} \alpha_i R_{\varphi_i}, \varphi_i \in S^2(V), \alpha_i \in \mathbb{R}\}, \quad \text{and}$$

$$\nu(n) = \max\{\nu(R) : R \in \mathcal{A}(V)\}.$$

In this way, we have our first invariant of algebraic curvature tensors. If we are given the values of $R, S \in \mathcal{A}(V)$ on two different bases, how can we distinguish them? Creating invariants seems to give more structure and delineation of the space of algebraic curvature tensors. If $\nu(R) \neq \nu(S)$, we can be sure that R and S are not the same tensor in a fundamental way. Investigating the behavior of $\nu(R)$ and $\nu(n)$ seems illuminating of the structure of algebraic curvature tensors and aims to better understand their invariants. In the next section, we investigate these invariants and their relation to the canonical algebraic curvature tensors in $\mathcal{A}(V)$.

2 Denseness of Canonical Algebraic Curvature Tesors

It was, in fact, shown in [2] that $\nu(3) = 2$. This is much lower than $\dim(\mathcal{A}(V))$, which is 6 when $\dim(V) = 3$. How can it be that for each element of $\mathcal{A}(V)$, we require only two canonical algebraic curvature tensors to build it? It seems that the canonical algebraic curvature tensors are highly prevalent in $\mathcal{A}(V)$ when $\dim(V) = 3$; they seem to live almost everywhere inside the space. Thus, we naturally ask: are the canonical algebraic curvature tensors dense in $\mathcal{A}(V)$ when $\dim(V) = 3$?

Theorem 1. Let V be a vector space of dimension 3. Then $\{\pm R_\varphi : \varphi \in S^2(V)\}$ is dense in $\mathcal{A}(V)$.

We present a lemma, one whose nature sheds light on the structure of $\mathcal{A}(V)$ when $\dim(V) = 3$, that will simplify the proof of our theorem.

Lemma 1. Let V be a vector space of dimension 3, and let $R \in \mathcal{A}(V)$. If on some basis $\{e_i\}$ of V, the only possible nonzero entries of R are R_{1221}, R_{1331}, and R_{2332}, and those entries are all in fact nonzero, then $R = \pm R_\varphi$ for $\varphi \in S^2(V)$.

Proof. Let V be a vector space of dimension 3, and let $R \in \mathcal{A}(V)$ such that

$$R_{1221} = a_{12} \neq 0$$
$$R_{1331} = a_{13} \neq 0$$
$$R_{2332} = a_{23} \neq 0$$

are the only nonzero entries of R on the basis $\{e_i\}$ of V. Then on this basis, let

$$\varphi = \begin{bmatrix} a_{13}\sqrt{\frac{a_{12}}{a_{13}a_{23}}} & 0 & 0 \\ 0 & \sqrt{\frac{a_{12}a_{23}}{a_{13}}} & 0 \\ 0 & 0 & \sqrt{\frac{a_{13}a_{23}}{a_{12}}} \end{bmatrix},$$

which we denote by $\mathrm{diag}(a_{13}\sqrt{\frac{a_{12}}{a_{13}a_{23}}}, \sqrt{\frac{a_{12}a_{23}}{a_{13}}}, \sqrt{\frac{a_{13}a_{23}}{a_{12}}})$.

Case 1: If all of the a_{ij} are positive, then the entries of φ are all real numbers, and it is easy check that $R_{ijji} = (R_\varphi)_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R = R_\varphi$.

Case 2: If two of the a_{ij} are negative (and the other positive), then the quantities are again all real numbers, and permute the basis (permuting the entries of φ as necessary) so that $a_{12}, a_{13} < 0$ and $a_{23} > 0$. It is straightforward to check that $R_{ijji} = (R_\varphi)_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R = R_\varphi$.

Case 3: However, if one of the a_{ij} is negative (and the others positive), then the entries of φ are not real numbers. In this case, permute the basis so that $a_{12}, a_{13} > 0$ and $a_{23} < 0$. Then in each entry of φ, replace a_{ij} by $-a_{ij}$ for all $i, j \in \{1, 2, 3\}$ distinct. It is then straightforward to check that $(R_\varphi)_{ijji} = -R_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R = -R_\varphi$.

Case 4: Finally, if all of the a_{ij} are negative, then the entries of φ are not real numbers, so we replace a_{ij} by $-a_{ij}$ for all $i, j \in \{1, 2, 3\}$ distinct in each entry of φ, and one checks that $(R_\varphi)_{ijji} = -R_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R = -R_\varphi$.

Thus, in all cases, $R = \pm R_\varphi$, so as long as $R_{ijji} \neq 0$ for $i, j \in \{1, 2, 3\}$ distinct are the only nonzero entries of R, then $R = \pm R_\varphi$ for $\varphi \in S^2(V)$. □

We are now ready to prove Theorem 1. We will induce a topology on $\mathcal{A}(V)$ determined by an inner product on V. To see this, note that $\mathcal{A}(V) \subset \bigotimes^4 V^*$. Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Let $\{e_i\}$ be an orthonormal basis of V, and let $\{e^i\}$ be the corresponding orthonormal (dual) basis of V^* under this inner product.

Then $\{e^i \otimes e^j \otimes e^k \otimes e^k\}$ is an orthonormal basis for $\bigotimes^4 V^*$. Then on this basis, for each $R \in \mathcal{A}(V)$, $R = \sum R_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^k$. Thus, $\langle R, R\rangle = \|R\|^2 = \sum R_{ijkl}^2$ for all $R \in \mathcal{A}(V)$. The ε-balls $B_\varepsilon(R) = \{S \in \mathcal{A}(V) : \|R - S\| < \varepsilon\}$ for $R \in \mathcal{A}(V)$ are the open sets in a basis for our topology on $\mathcal{A}(V)$. Thus, we must show that for all $R \neq \pm R_\varphi$ for any $\varphi \in S^2(V)$, there exists $\pm R_\varphi \in B_\varepsilon(R)$ for all $\varepsilon > 0$.

Proof. (Theorem 1) Let V be a vector space of dimension 3, and let $R \in \mathcal{A}(V)$ such that $R \neq \pm R_\varphi$ for any $\varphi \in S^2(V)$. Note that $R \not\equiv 0$, as $0 = R_\varphi$, where $\varphi \equiv 0$. Let $\varepsilon > 0$. Klinger showed in [5] that for all $R \in \mathcal{A}(V)$, there exists an orthonormal basis $\{f_i\}$ such that R_{ijji} for distinct $i, j \in \{1, 2, 3\}$ are the only possible nonzero entries of R. By Lemma 1, it must be the case that $R_{k\ell\ell k} = 0$ for some distinct $k, \ell \in \{1, 2, 3\}$. Then define $S \in \mathcal{A}(V)$ such that $S_{ijji} = R_{ijji}$ for all $i, j \in \{1, 2, 3\}$ distinct with $R_{ijji} \neq 0$, and let $S_{k\ell\ell k} = \frac{\varepsilon}{2}$ for any $k, \ell \in \{1, 2, 3\}$ distinct such that $R_{k\ell\ell k} = 0$. If there is one such $R_{k\ell\ell k} = 0$, then

$$\langle R - S, R - S\rangle = \|R - S\|^2 = \sum_{i,j \in \{1,2,3\}, i \neq j} (R - S)_{ijji}^2 = \left(-\frac{\varepsilon}{2}\right)^2 = \frac{\varepsilon^2}{4}.$$

Thus, $\|R - S\| = \frac{\varepsilon}{2} < \varepsilon$. Thus, $S \in B_\varepsilon(R)$.
If there are two such $R_{k\ell\ell k} = 0$, then

$$\langle R - S, R - S\rangle = \|R - S\|^2 = \sum_{i,j \in \{1,2,3\}, i \neq j} (R - S)_{ijji}^2 = \left(-\frac{\varepsilon}{2}\right)^2 + \left(-\frac{\varepsilon}{2}\right)^2 = \frac{\varepsilon^2}{2}.$$

Thus, $\|R - S\| = \frac{\varepsilon}{\sqrt{2}} < \varepsilon$. Thus, $S \in B_\varepsilon(R)$.
In either case, we find $S \in B_\varepsilon(R)$ with S_{ijji} nonzero for $i, j \in \{1, 2, 3\}$ distinct and only nonzero on those entries, so by Lemma 1, $S = \pm R_\varphi$ for some $\varphi \in S^2(V)$. Thus, for all $R \in \mathcal{A}(V)$, for all $\varepsilon > 0$, there exists $\pm R_\varphi \in B_\varepsilon(R)$, $\varphi \in S^2(V)$. This proves that every open set in $\mathcal{A}(V)$ contains some element of $\{\pm R_\varphi : \varphi \in S^2(V)\}$. Thus, $\{\pm R_\varphi : \varphi \in S^2(V)\}$ is dense in $\mathcal{A}(V)$. ☐

Corollary 1. *(of Theorem 1)* If $\dim(V) = 3$, the set

$$\{\pm R_\varphi : \varphi \in S^2(V), \mathrm{Rank}(\varphi) = 3\}$$

is dense in $\mathcal{A}(V)$. In our proof of Theorem 1, each symmetric bilinear form that we create to prove the existence of a canonical algebraic curvature tensor has rank 3.

This result sheds light on the structure of canonical algebraic curvature tensors. It could, perhaps, be useful to realize certain algebraic curvature tensors as the limits of sequences of canonical algebraic curvature tensors.

Of course, we naturally ask: are the canonical algebraic curvature tensors dense in $\mathcal{A}(V)$ when $\dim(V) > 3$? As the dimension of $S^2(V)$ is quadratic in $\dim(V) = n$, and the dimension of $\mathcal{A}(V)$ is quartic in n, it hardly seems possible that we could realize the smooth image of $S^2(V)$ into $\mathcal{A}(V)$ as a dense subset. However, could collections of more than one canonical algebraic curvature tensor be dense in $\mathcal{A}(V)$ in higher dimensions? We make the following definition.

Definition 6. We define

$$U_k = \{R \in \mathcal{A}(V) : \nu(R) \leq k\}.$$

Thus, we ask: for which values of k is U_k dense in $\mathcal{A}(V)$? If $\nu(n) = k$ for $\dim(V) = n$, it would seem that those $R \in \mathcal{A}(V)$ for which $\nu(R) = k - 1$ would be highly prevalent in $\mathcal{A}(V)$. This motivates the following conjecture.

Conjecture 1. If $\dim(V) = n$ and $\nu(n) = k$, then U_{k-1} is dense in $\mathcal{A}(V)$.

If one could find the relationship between $\nu(n)$ and the denseness of U_k for some k, one could put sharper bounds on $\nu(n)$, or if one were to prove Conjecture 1 and find computational ways of verifying the denseness of collections of canonical algebraic curvature tensors, sharper information about the value and behavior of $\nu(n)$ could be revealed. Thus, asking questions about denseness of collections of canonical algebraic curvature tensors aids in both the quest to investigate invariants and the questions about the structure of canonical algebraic curvature tensors. In the next section, we investigate the invariants of algebraic curvature tensors in a different manner.

3 The Signature Conjecture

Since for all $\alpha \in \mathbb{R}$, $\alpha R_\varphi = \varepsilon R_{\sqrt{|\alpha|}\varphi}$ for $\varepsilon = \pm 1$, asking about the (minimal) expression $R = \sum_{i=1}^{\nu(R)} \alpha_i R_{\varphi_i}$ for $\alpha_i \in \mathbb{R}$ and $\varphi_i \in S^2(V)$ is equivalent to asking about the expression $R = \sum_{i=1}^{\nu(R)} \varepsilon_i R_{\tilde{\varphi}_i}$ for $\varepsilon_i = \pm 1$ and $\tilde{\varphi}_i \in S^2(V)$.

In [1], it was shown that if $\mathrm{Rank}(\varphi) \geq 3$, then there does not exist $\psi \in S^2(V)$ such that $R_\varphi = -R_\psi$. With this in mind, the following conjecture was made.

Conjecture 2. (The Signature Conjecture) Let $R \in \mathcal{A}(V)$. For any minimal expression

$$R = \sum_{i=1}^{\nu(R)} \varepsilon_i R_{\varphi_i}$$

with $\mathrm{Rank}(\varphi_i) \geq 3$ for all i, the number of indices i for which $\varepsilon_i = -1$ is unique.

This conjecture aims to construct an invariant (the number of $\varepsilon_i = -1$ in a minimal expression) of algebraic curvature tensors. Requiring minimality of the expression in the conjecture seems natural, preventing obvious counterexamples, and the rank condition on the φ_i prevents the counterexamples $R_\varphi = -R_\psi$ for certain φ and ψ of rank 2.

However, it was shown in [6] that for any $\tau \in S^2(V)$ of rank $k-1$ for $2 \leq k-1 \leq n-1$, there exist $\varphi_1, \varphi_2, \psi_1, \psi_2 \in S^2(V)$ of rank k such that $R_\tau = R_{\varphi_1} + R_{\varphi_2} = R_{\psi_1} - R_{\psi_2}$, providing counterexamples to the signature conjecture. The work done in that text attempted to reformulate the signature conjecture with requirements on the ranks of the forms involved to work around these counterexamples, but in these cases where $R_{\varphi_1} + R_{\varphi_2} = R_{\psi_1} - R_{\psi_2}$ in [6], it was always the case that the signatures of the forms involved differed. This indicates a different way to revise the signature conjecture: instead of making requirements just on the ranks of the forms involved, can we require that the forms be of a certain signature? To be able to state this new version of the signature conjecture, we must first prove that the algebraic curvature tensors created from symmetric bilinear forms of a chosen signature span the space of algebraic curvature tensors. Unfortunately, we do not have such a general result, but we present such a result for forms of rank less than or equal to 3. Before we state this result, we should say a word about how we think of different signatures, and we state a few lemmas.

3.1 A Signature-Driven Revision to the Signature Conjecture

We aim to express algebraic curvature tensors as linear combinations of canonical algebraic curvature tensors from forms of a certain signature. In doing so, we remember that if the signature of φ is (p, q, s), then the signature of $-\varphi$ is (q, p, s), and $R_\varphi = R_{-\varphi}$. Thus, to eliminate redundancy, we wish to treat forms of signature (p, q, s) and forms of signature (q, p, s) as the same. Thus, we make the following definition.

Definition 7. For $\varphi \in S^2(V)$ of signature (p, q, s), we define the *adjusted signature* of φ, denoted $\operatorname{sgn}(\varphi)$, as

$$\operatorname{sgn}(\varphi) = \begin{cases} (p, q, s) & \text{if } p \leq q \\ (q, p, s) & \text{if } p > q. \end{cases}$$

From now on, we will only consider the adjusted signature of bilinear forms.

We now cite a few lemmas that will aid our proof of the spanning nature of canonical algebraic curvature tensors built from forms of a certain signature.

Lemma 2. [7] Let $\varphi \in S^2(V)$ with $\operatorname{sgn}(\varphi) = (p, q, s)$. If $p = 0$ and $q \geq 2$ (the $q = 1$ case is trivial; this would imply $R_\varphi = 0$), then

$$R_\varphi = \sum_i \pm R_{\tau_i}$$

with $\operatorname{sgn}(\tau_i) = (0, 2, n-2)$ for all i. And if $p \geq 1$, then

$$R_\varphi = \sum_i \pm R_{\psi_i}$$

with $\operatorname{sgn}(\psi_i) = (1, 1, n-2)$ for all i.

Lemma 3. [6] Let $\varphi \in S^2(V)$ with $\text{sgn}(\varphi) = (p, q, s)$, where $s \geq 1$. Then

$$R_\varphi = R_{\tau_1} + R_{\tau_2}$$

with $\text{sgn}(\tau_1) = (p+1, q, s-1)$ and $\text{sgn}(\tau_2) = (p, q+1, s-1)$.

We are now ready to state our theorems. We hope to extend these theorems to symmetric bilinear forms of any rank.

Theorem 2.

$$\mathcal{A}(V) = \text{span}\{R_\varphi : \varphi \in S^2(V),\ \text{sgn}(\varphi) = (p, q, s)\}$$

for any adjusted signature (p, q, s), where $\text{Rank}(\varphi) = 2$.

Proof. Clearly, any linear combination of canonical algebraic curvature tensors is itself an algebraic curvature tensor, so

$$\text{span}\{R_\varphi : \varphi \in S^2(V),\ \text{sgn}(\varphi) = (p, q, s)\} \subset \mathcal{A}(V)$$

for all adjusted signatures (p, q, s).

Let $R \in \mathcal{A}(V)$. Write $R = \sum_{i=1}^{\nu(R)} \pm R_{\varphi_i}$ for $\varphi_i \in S^2(V)$. Use Lemma 2 to express

$$R = \sum_{i=1}^{\nu(R)} \pm R_{\varphi_i} = \sum_j \pm R_{\tau_j} + \sum_k \pm R_{\psi_k},$$

where $\text{sgn}(\tau_j) = (1, 1, n-2)$ and $\text{sgn}(\psi_k) = (0, 2, n-2)$ for all j and k. $\text{Rank}(\varphi) = 2$, so $p + q = 2$. We show that if $\text{sgn}(\tau) = (1, 1, n-2)$, then R_τ can be written as $R_\tau = -R_\gamma$, where $\text{sgn}(\gamma) = (0, 2, n-2)$.

Let $\tau \in S^2(V)$ with $\text{sgn}(\tau) = (1, 1, n-2)$. Then by the Spectral Theorem, τ is diagonal on some basis $\{e_i\}$ of V. We can permute and rescale this basis so that $\tau = \text{diag}(-1, 1, 0, ..., 0)$. Then on this basis, the only possible nonzero curvature entry of R_τ is $(R_\tau)_{1221}$. Then on this basis, let $\gamma = \text{diag}(1, 1, 0, ..., 0)$. Clearly, $(R_\tau)_{1221} = -1 = -(R_\gamma)_{1221}$ Thus, $R_\tau = -R_\gamma$ with $\text{sgn}(\gamma) = (0, 2, n-2)$. This (by the same reasoning) additionally shows that if $\gamma \in S^2(V)$ with $\text{sgn}(\gamma) = (0, 2, n-2)$, then R_γ can be written as $R_\gamma = -R_\tau$, where $\text{sgn}(\tau) = (1, 1, n-2)$. Thus, we can use either of these processes on each τ_j, ψ_k to write

$$R = \sum_{i=1}^{\nu(R)} \pm R_{\varphi_i} = \sum_j \pm R_{\tau_j} + \sum_k \pm R_{\psi_k} = \sum_\ell \pm R_{\eta_\ell},$$

where $\text{sgn}(\eta_\ell) = (p, q, s)$ with $p + q = 2$ for all ℓ. Thus, $R \in \text{span}\{R_\varphi : \varphi \in S^2(V),\ \text{sgn}(\varphi) = (p, q, s)\}$ for all adjusted signatures (p, q, s) with $p + q = 2$. Thus, $\mathcal{A}(V) \subset \text{span}\{R_\varphi : \varphi \in S^2(V),\ \text{sgn}(\varphi) = (p, q, s)\}$ for all adjusted signatures (p, q, s) with $\text{Rank}(\varphi) = 2$. Thus,

$$\mathcal{A}(V) = \text{span}\{R_\varphi : \varphi \in S^2(V),\ \text{sgn}(\varphi) = (p, q, s)\}$$

for any adjusted signature (p, q, s), where $\text{Rank}(\varphi) = 2$. $\square$

Theorem 3.

$$\mathcal{A}(V) = \operatorname{span}\{R_\varphi : \varphi \in S^2(V),\ \operatorname{sgn}(\varphi) = (p, q, s)\}$$

for any adjusted signature (p, q, s), where $\operatorname{Rank}(\varphi) = 3$.

Proof. $\operatorname{Rank}(\varphi) = 3$, so $p + q = 3$. Use the result of Theorem 2 to write $R = \sum_\ell \pm R_{\eta_\ell}$, where $\operatorname{sgn}(\eta_\ell) = (1, 1, n-2)$ for all ℓ. Use Lemma 3 on each η_ℓ to write $R = \sum_m \pm R_{\gamma_m}$, where $\operatorname{sgn}(\gamma_m) = (1, 2, n-3)$ or $(2, 1, n-3)$ for all m, and we remember that $(1, 2, n-3)$ and $(2, 1, n-3)$ are both identified with the adjusted signature $(1, 2, n-3)$. Thus, $R = \sum_m \pm R_{\gamma_m}$, where $\operatorname{sgn}(\gamma_m) = (1, 2, n-3)$ for all m.

It remains to show that if $\gamma \in S^2(V)$ with $\operatorname{sgn}(\gamma) = (1, 2, n-2)$, then we can express R_γ as $R_\gamma = \sum_r \pm R_{\theta_r}$ with $\operatorname{sgn}(\theta_r) = (0, 3, n-3)$ for all r.

Let $\gamma \in S^2(V)$ with $\operatorname{sgn}(\gamma) = (1, 2, n-3)$. Then on some basis of V, $\gamma = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, 0, ..., 0)$, where $\lambda_1, \lambda_2 > 0$ and $\lambda_3 < 0$. Then on this basis,

$$\begin{aligned} -(R_\gamma)_{1221} &= -\lambda_1\lambda_2 = a < 0 \\ -(R_\gamma)_{1331} &= -\lambda_1\lambda_3 = b > 0 \\ -(R_\gamma)_{2332} &= -\lambda_2\lambda_3 = c > 0 \end{aligned}$$

are the only nonzero curvature entries of $-R_\gamma$.

Case 1: $c > 1$. On this basis that diagonalizes γ, let $\theta_1 = \operatorname{diag}(\frac{b-a}{c-1}, 1, c, 0, ..., 0)$, $\theta_2 = \operatorname{diag}(\frac{ca-b}{c-1} + z, 1, 1)$, and $\theta_3 = \operatorname{diag}(z, 1, 1)$, where we pick $z > 0$ sufficiently large such that $z > -\frac{ca-b}{c-1}$. One can check that $-(R_\gamma)_{ijji} = (R_{\theta_1})_{ijji} + (R_{\theta_2})_{ijji} - (R_{\theta_3})_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R_\gamma = \sum_r \pm R_{\theta_r}$ with $\operatorname{sgn}(\theta_r) = (0, 3, n-3)$ for all r.

Case 2: $c = 1$. On the same basis, let $\theta_1 = \operatorname{diag}(\frac{-a}{3} + \frac{4b}{3} - \frac{5z}{9}, 1, 1, 0, ..., 0)$, $\theta_2 = (\frac{2a}{3} - \frac{2b}{3} + \frac{16z}{9}, 2, \frac{1}{2}, 0, ..., 0)$, and $\theta_3 = \operatorname{diag}(z, 3, \frac{1}{3}, 0, ..., 0)$, where we pick $z > 0$ such that $\frac{3}{5}(4b - a) > z > \frac{3}{8}(b - a)$, so that all entries of the θ_r are positive, and note that this choice of z is always possible as $\frac{3b}{b-a} + 1 > 1$, so $\frac{4b-a}{b-a} > 1$, which implies that $\frac{8}{5}\frac{4b-a}{b-a} > 1$, implying that $\frac{3}{5}(4b - a) > \frac{3}{8}(b - a)$ for all values of a and b. One can check that $-(R_\gamma)_{ijji} = (R_{\theta_1})_{ijji} + (R_{\theta_2})_{ijji} - (R_{\theta_3})_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R_\gamma = \sum_r \pm R_{\theta_r}$ with $\operatorname{sgn}(\theta_r) = (0, 3, n-3)$ for all r.

Case 3: $0 < c < 1$. On the same basis, let $\theta_1 = \operatorname{diag}(\frac{b-ac^2}{c(1-c)}, 1, c, 0, ..., 0)$, $\theta_2 = \operatorname{diag}(\frac{ca-b}{1-c} + z, \frac{1}{c}, c, 0, ..., 0)$, and $\theta_3 = \operatorname{diag}(z, \frac{1}{c}, c, 0, ..., 0)$, where we pick $z > 0$ sufficiently large such that $z > -\frac{ca-b}{1-c}$. One can check that $-(R_\gamma)_{ijji} = (R_{\theta_1})_{ijji} + (R_{\theta_2})_{ijji} - (R_{\theta_3})_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R_\gamma = \sum_r \pm R_{\theta_r}$ with $\operatorname{sgn}(\theta_r) = (0, 3, n-3)$ for all r.

Thus, in all cases, $R_\gamma = \sum_r \pm R_{\theta_r}$ with $\operatorname{sgn}(\theta_r) = (0, 3, n-3)$ for all r. We can use this process on each γ_m to write

$$R = \sum_m \pm R_{\gamma_m} = \sum_t \pm R_{\delta_t},$$

where $\operatorname{sgn}(\delta_t) = (0, 3, n-3)$ for all t. Thus, $R \in \operatorname{span}\{R_\varphi : \varphi \in S^2(V),\ \operatorname{sgn}(\varphi) = (p, q, s)\}$ for all adjusted signatures (p, q, s) with $p + q = 3$. Thus, $\mathcal{A}(V) \subset \operatorname{span}\{R_\varphi : \varphi \in S^2(V),\ \operatorname{sgn}(\varphi) = (p, q, s)\}$ for all adjusted signatures (p, q, s) with $p + q = 3$.

Thus,

$$\mathcal{A}(V) = \operatorname{span}\{R_\varphi : \varphi \in S^2(V),\ \operatorname{sgn}(\varphi) = (p, q, s)\}$$

for all adjusted signatures (p, q, s) with $\operatorname{Rank}(\varphi) = 3$. $\square$

We may now define some new invariants of algebraic curvature tensors, whose properties are interesting in their own right and warrant meaningful investigation, that will aid in the statement of our revision to the signature conjecture.

Definition 8. Let V be a vector space of dimension n, and let $R \in \mathcal{A}(V)$. We define the quantities

$$\nu_{(p,q,s)}(R) = \min\{k : R = \sum_{i=1}^{k} \pm R_{\varphi_i}, \varphi_i \in S^2(V),\ \operatorname{sgn}(\varphi_i) = (p, q, s)\}$$

$$\nu_{(p,q,s)}(n) = \max\{\nu_{(p,q,s)}(R) : R \in \mathcal{A}(V)\}.$$

Note that according to the results we have, these quantities are only well-defined for adjusted signatures of forms of rank less than or equal to 3.

With our new quantities defined, we may state our new version of the signature conjecture. However, we first state one result of the investigation of $\nu_{(p,q,s)}$.

Theorem 4. $\nu_{(0,3)}(3) \leq 3$.

Proof. Let V be a vector space of dimension 3, and let $R \in \mathcal{A}(V)$. By [5], there exists a basis $\{e_i\}$ of V such that the only possible nonzero entries of R are R_{ijji} for $i, j \in \{1, 2, 3\}$ distinct. We work through the cases of the signs of these entries. Throughout, let $R_{ijji} = a_{ij}$.

Case 1: $a_{ij} > 0$ for all $i, j \in \{1, 2, 3\}$ distinct. Use Case 1 in the proof of Lemma 1 to write $R = R_\varphi$ and notice that $\operatorname{sgn}(\varphi) = (0, 3)$.

Case 2: $a_{ij} < 0$ for all $i, j \in \{1, 2, 3\}$ distinct. Use the method of Case 4 in the proof of Lemma 1, replacing $a_{13}\sqrt{\frac{a_{12}}{a_{13}a_{23}}}$ by $\sqrt{\frac{a_{12}a_{13}}{a_{23}}}$ to write $R = -R_\varphi$ and notice again that $\operatorname{sgn}(\varphi) = (0, 3)$.

Case 3: Without loss of generality, $a_{12} < 0$ and $a_{13}, a_{23} > 0$. In the proof of Theorem 3, we showed precisely that if these assumptions on the R_{ijji} entries are satisfied, then $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 4: Without loss of generality, $a_{12} > 0$ and $a_{13}, a_{23} < 0$. It is a simple matter to replace R by $-R$ and then use the result of the previous case to show that $R = \sum_{k-1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 5: Without loss of generality, $a_{12} = 0$ and $a_{13}, a_{23} > 0$. If $a_{23} > 1$, use the construction of Case 1 of Theorem 3 to express $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k. If $a_{23} = 1$, use Case 2 of Theorem 3 to write $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k. If $0 < a_{23} < 1$, use the construction of Case 3 of Theorem 3 to write $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 6: Without loss of generality, $a_{12} = 0$ and $a_{13}, a_{23} < 0$. It is a simple matter to replace R by $-R$ and use the result of the previous case to show that $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 7: Without loss of generality, $a_{12} = 0$, $a_{13} < 0$, and $a_{23} > 0$. If $a_{23} > 1$, use the construction of Case 3 of Theorem 3 to write $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k. If $a_{23} = 1$, let $\varphi_1 = \operatorname{diag}(\frac{1}{14} z - \frac{1}{7} a_{13}, 1, 2)$, $\varphi_2 = \operatorname{diag}(\frac{3}{7} a_{13} + \frac{9}{7} z, \frac{1}{3}, 3)$, and $\varphi_3 = \operatorname{diag}(z, \frac{1}{2}, 4)$, where we pick $z > 0$ such that $\frac{3}{7} a_{13} + \frac{9}{7} z > 0$, and check that $R_{ijji} = (R_{\varphi_1})_{ijji} + (R_{\varphi_2})_{ijji} - (R_{\varphi_3})_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Notice $\operatorname{sgn}(\varphi_i) = (0, 3)$ for all i. If $0 < a_{23} < 1$, use the construction of Case 1 of Theorem 3 to write $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$ with $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 8: Without loss of generality, $a_{12} = 0$, $a_{13} = 0$, and $a_{23} > 0$. Let $\varphi_1 = \operatorname{diag}(1, 2, \frac{1}{2}(z - a_{23}))$, $\varphi_2 = \operatorname{diag}(a_{23} + z, \frac{1}{3}, 3)$, and $\varphi_3 = \operatorname{diag}(1, 3, z)$, where we pick $z > a_{23} > 0$. One can check that $R_{ijji} = (R_{\varphi_1})_{ijji} + (R_{\varphi_2})_{ijji} - (R_{\varphi_3})_{ijji}$ for all distinct $i, j \in \{1, 2, 3\}$. Thus, $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$, where $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Case 9: Without loss of generality, $a_{12} = 0, a_{13} = 0$, and $a_{23} < 0$. It is a simple matter to replace R by $-R$ and use the result of the previous case to write $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$, where $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k.

Thus, by examining every possibility of the signs of the values of R, we examine every possible collection of values of R. In all cases, $R = \sum_{k=1}^{3} \pm R_{\varphi_k}$, where $\operatorname{sgn}(\varphi_k) = (0, 3)$ for all k or $R = \pm R_{\varphi}$ with $\operatorname{sgn}(\varphi) = (0, 3)$ for all $R \in \mathcal{A}(V)$. Thus, $\nu_{(0,3)}(3) \leq 3$. $\square$

Conjecture 3. (The Incomplete Adjusted Signature Conjecture) Let $R \in \mathcal{A}(V)$. For any minimal expression

$$R = \sum_{i=1}^{\nu_{(p,q,s)}(R)} \varepsilon_i R_{\varphi_i}$$

with $\operatorname{Rank}(\varphi_i) \leq 3$ and $\operatorname{sgn}(\varphi_i) = (p, q, s)$ for all i, the number of indices i for which $\varepsilon_i = -1$ is unique.

This conjecture is well-posed, but for the sake of the clarity of our goals, we will also state the version of the signature conjecture that we would like to posit.

Conjecture 4. (The Adjusted Signature Conjecture) Let $R \in \mathcal{A}(V)$. For any minimal expression

$$R = \sum_{i=1}^{\nu_{(p,q,s)}(R)} \varepsilon_i R_{\varphi_i}$$

with $\text{sgn}(\varphi_i) = (p,q,s)$ for all i, the number of indices i for which $\varepsilon_i = -1$ is unique.

This conjecture is not yet well-posed, but we can, in fact, prove that it is true in the most minimal case.

Theorem 5. Let $R \in \mathcal{A}(V)$. If $\nu_{(p,q,s)}(R) = 1$ for some adjusted signature (p,q,s), then in any minimal expression

$$R = \varepsilon R_\varphi$$

with $\varepsilon = \pm 1$ and $\text{sgn}(\varphi) = (p,q,s)$, ε is unique.

Proof. Let $R \in \mathcal{A}(V)$ with $\nu_{(p,q,s)}(R) = 1$ for some signature (p,q,s). Assume for contradiction that $R = R_\varphi = -R_\psi$ for $\varphi, \psi \in S^2(V)$ with $\text{sgn}(\varphi) = \text{sgn}(\psi) = (p,q,s)$. Then by [1], it must be the case that $\text{Rank}(\varphi) = \text{Rank}(\psi) = 2$. Then on some basis $\{e_i\}$ of V, $\varphi = \text{diag}(\pm 1, \pm 1, 0, ..., 0)$. It is simple to prove that for any $\varphi \in S^2(V)$, $\ker(\varphi) = \ker(R_\varphi) = \{x \in V : R(x,y,z,w) = 0 \text{ for all } y,z,w \in V\}$. Thus, $\ker(\varphi) = \ker(R_\varphi) = \ker(-R_\psi) = \ker(R_\psi) = \ker(\psi)$, so the only nonzero entries of $\psi = [a_{ij}] = \psi(e_i, e_j)$ on the basis $\{e_i\}$ are $a_{11}, a_{12} = a_{21}$, and a_{22}. Thus, we can say that there exists another basis $\{f_i\}$ on which $\psi = \text{diag}(\pm 1, \pm 1, 0, ..., 0)$, where $f_1 = \alpha e_1 + \beta e_2$ and $f_2 = \gamma e_1 + \delta e_2$ for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. One can check that for any algebraic curvature tensor $R \in \mathcal{A}(V)$,

$$\begin{aligned} R(f_1, f_2, f_2, f_1) &= R(\alpha e_1 + \beta e_2, \gamma e_1 + \delta e_2, \gamma e_1 + \delta e_2, \alpha e_1 + \beta e_2) \\ &= (\alpha\delta - \beta\gamma)^2 R(e_1, e_2, e_2, e_1) = (\det(A))^2 R(e_1, e_2, e_2, e_1), \end{aligned}$$

where A is the change of basis matrix $\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$ from $\{e_i : i = 1, 2\}$ to $\{f_i : i = 1, 2\}$. Thus,

$$\begin{aligned} R_\psi(f_1, f_2, f_2, f_1) &= (\det(A))^2 R_\psi(e_1, e_2, e_2, e_1) \\ &= -(\det(A))^2 R_\varphi(e_1, e_2, e_2, e_1) \end{aligned}$$

And we divide this equation by $R_\psi(f_1, f_2, f_2, f_1) = R_\varphi(e_1, e_2, e_2, e_1)$, as $\text{sgn}(\varphi) = \text{sgn}(\psi)$, on both sides to get

$$1 = -(\det(A))^2,$$

so $(\det(A))^2 = -1$. This contradicts that $\det(A) \in \mathbb{R}$. Hence, our assumption must be false. Thus, it is not the case that $R = R_\varphi = -R_\psi$ for any $\varphi, \psi \in S^2(V)$ with $\text{sgn}(\varphi) = \text{sgn}(\psi) = (p,q,s)$. This proves that in any minimal expression

$$R = \varepsilon R_\varphi$$

with $\varepsilon = \pm 1$ and $\text{sgn}(\varphi) = (p,q,s)$, ε is unique. $\square$

These are the hopeful beginnings of the construction of invariants of algebraic curvature tensors. While our desired conjecture is not yet well-posed, we prove it in its simplest case, and we pose a weaker version of the conjecture whose proof seems by no means out of reach.

4 Open Questions

4.1 Denseness of Canonical Algebraic Curvature Tensors

1. For which values of k is U_k dense in $\mathcal{A}(V)$?

2. Can a converse to Conjecture 1 be proven, or any statement similar to it? Could it be shown that U_{k-1} being dense in $\mathcal{A}(V)$ implies that $\nu(n) = k$? Proving such a statement and computationally verifying denseness of collections of canonical algebraic curvature tensors would further determine the values of $\nu(n)$. Does the denseness of a certain U_k put a bound on $\nu(n)$?

3. Could U_k for some value of k be realized as a submanifold of $\mathcal{A}(V)$? Knowing about the characterization of dense submanifolds could aid in the determination of the denseness of U_k.

4. Is U_1 dense in U_2 when $\dim(V) > 3$? Is U_{k-1} ever dense in U_k for any k?

5. Is $\{\pm R_\varphi : \varphi \in S^2(V), \operatorname{sgn}(\varphi) = (0,3)\}$ dense in $\mathcal{A}(V)$ when $\dim(V) = 3$? Are the canonical algebraic curvature tensors (or collections of them) built from forms of a certain signature dense in $\mathcal{A}(V)$ in any dimension?

6. Einstein tensors - those for which the Ricci map is a multiple of the metric on the manifold at hand - seem to have a close relationship to the canonical algebraic curvature tensors of symmetric build. Are the canonical algebraic curvature tensors (or collections of them) dense in the Einstein tensors?

4.2 The Adjusted Signature Conjecture

1. What are computational or constructive ways to verify that $\nu(R) = k$ for some R? Can this be done for $\nu_{(p,q,s)}(R)$ as well? Can sharp bounds be determined for this invariant? Constructing an example for which $\nu_{(0,3)}(R) = 3$ would prove that $\nu_{(0,3)}(3) = 3$.

2. Can minimal expressions of algebraic curvature tensors, whether making requirements on the adjusted signatures of the forms involved or not, be characterized based on information known about the tensors at hand? Do the curvature entries on a basis tell us something about the forms from which the minimal sum of canonical algebraic curvature tensors is built? Can computational methods be developed for verifying that a certain expression is minimal?

3. One could use Lemma 3 to show that if $\text{sgn}(\tau) = (0, 3, n-3)$, then $R_\tau = R_{\varphi_1} + R_{\varphi_2}$, where $\text{sgn}(\varphi_1) = (1, 3, n-4), \text{sgn}(\varphi_2) = (0, 4, n-4)$. Can it be shown that if $\text{sgn}(\varphi) = (1, k-1, n-k)$, then $R_\varphi = \sum \pm R_{\psi_i}$ with $\text{sgn}(\psi_i) = (0, k, n-k)$? Combining this with Lemma 3 would prove that the canonical algebraic curvature tensors built from positive-definite symmetric bilinear forms span the space of algebraic curvature tensors.

4. If $\text{sgn}(\varphi) = (p, q, s)$, can it be shown that $R_\varphi = \sum \pm R_{\psi_i}$ with $\text{sgn}(\psi_i) = (p+1, q-1, s)$? Proving this would prove that the canonical algebraic curvature tensors built from forms of any adjusted signature span the space of algebraic curvature tensors.

5. If the only nonzero entries of $R \in \mathcal{A}(V)$ are R_{ijji} for some i, j distinct on some orthonormal basis, then R is considered *pure*. If R is pure, are the forms in its minimal expression in terms of canonical algebraic curvature tensors simultaneously diagonalizable? How would one determine if a pure tensor is a canonical algebraic curvature tensor?

6. What is the structure of $U_k^{(p,q,s)}$, where $U_k^{(p,q,s)}$ is U_k requiring the adjusted signature of each form involved to be (p, q, s)? Is this set ever dense, or is it ever a submanifold of $\mathcal{A}(V)$? What does it look like inside $\mathcal{A}(V)$? Investigating these both topologically and geometrically would be interesting.

7. What is the relation between $R = \sum_{i}^{\nu(p,q,s)} \pm R_{\varphi_i}$ and $R = \sum_{j}^{\nu(m,n,r)} \pm R_{\psi_i}$ for $(p, q, s) \neq (m, n, r)$? Can relationships among the forms involved or the linear independence of the canonical algebraic curvature tensors be determined? Which adjusted signature uses the fewest canonical algebraic curvature tensors? Does this adjusted signature depend on R? If so, how?

8. If the minimal expressions of $R, S \in \mathcal{A}(V)$ use different numbers of $\varepsilon_i = -1$, can R and S be realized as the Riemann curvature tensor at different points on the same manifold?

9. Can any of the results of this work be shown for algebraic curvature tensors built from antisymmetric bilinear forms? That is, one can define $V_k = \{R \in \mathcal{A}(V) : R = \sum_{i=1}^{m} \pm R_{\psi_i}, \psi_i \in \Lambda^2(V), m \leq k\}$ and ask
   - Is V_1 dense in $\mathcal{A}(V)$ when $\dim(V) = 3$?
   - Is V_{k-1} ever dense in V_k?
   - Can computational methods be developed for determining denseness of a V_k? Does this put a bound on the number of canonical algebraic curvature tensors of antisymmetric build needed to make any algebraic curvature tensor?

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