On the Structure Groups of Decomposable Algebraic Curvature Tensors

Cole Franks

August 19, 2011

Abstract

This paper examines the action of $GL_N(\mathbb{R})$ on decomposable algebraic curvature tensors. The main result is that the structure group of a decomposable algebraic curvature tensor can only permute the subspaces into which the tensor decomposes: if the tensor decomposes into tensors R_i on V_i where $V = \bigoplus_{i=1}^{k} V_i$, then for any matrix A in the structure group there exists a permutation σ such that $A : V_i \to V_{\sigma(i)}$. Restrictions are found for when $\sigma(i) = j$; for one, $dim(V_i)$ must equal $dim(V_j)$. We find A restricted to V_i, namely if we set $W = V_i$ then the set of possible maps $A|_W : W \to V_j$ is isomorphic to a particular left coset of the structure group of R_i.

1 Introduction

A pseudo-Riemannian manifold is a real differentiable manifold M in which the tangent space $T_p(M)$ at each point p has a symmetric bilinear form g that varies smoothly from point to point [1]. An example of a pseudo-Riemannian manifold is the four dimensional space-time of our universe. There is a way to measure the curvature of the manifold at each point known as the Riemann curvature tensor, ${}^gR$. The tensor is of type $(0, 4)$ and has several symmetries in its arguments; for example it is skew-symmetric in the first two arguments and the last two arguments. A tensor R of type $(0, 4)$ on an arbitrary vector space is an algebraic curvature tensor if it satisfies the all the same symmetries as ${}^gR$. The set of all algebraic curvature tensors on an N dimensional vector space V is denoted $\mathcal{A}(V)$. Every algebraic curvature tensor can arise as the curvature tensor of pseudo-Riemannian manifold at a point [2], so algebraic curvature tensors are worthwhile objects of study.

Given two algebraic curvature tensors $R_1 \in \mathcal{A}(V_1)$ and $R_2 \in \mathcal{A}(V_2)$ such that $V_1 \oplus V_2 = V$, with $dim(V_1), dim(V_2) > 0$, one can build another curvature tensor R on V in a natural way. Define $R = R_1 \oplus R_2 \in \mathcal{A}(V)$ so that $R(x, y, z, w) = 0$ if one of x, y, z, or w is an element of V_1 and another one is an element of V_2. If $x, y, z, w \in V_1$ then $R(x, y, z, w) := R_1(x, y, z, w)$, and if $x, y, z, w \in V_2$ and $R(x, y, z, w) := R_2(x, y, z, w)$. Essentially,"crossterms" mixing arguments from

V_1 and V_2 all vanish, and the other terms are determined by R_1 and R_2 and the fact that R is multilinear. If an algebraic curvature tensor can be written this way, it is said to be decomposable.

$GL_N(\mathbb{R})$ acts on $\mathcal{A}(V)$ in a natural way by precomposition with the inverse of each element: if $A \in GL_N(\mathbb{R})$ and $R \in \mathcal{A}(V)$ then $A \cdot R(x, y, z, w) := R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w)$. We seek the structure group of R under the action of $GL_N, \mathbb{R})$ (the subgroup of $GL_N(\mathbb{R})$ with all elements leaving R fixed. In the language of group actions this is the isotropy subgroup. This paper generalizes the work in [3] to describe the structure group of decomposable algebraic curvature tensors in terms of the structure groups of their constituent curvature tensors.

2 Preliminary Material

Definition 1. A tensor $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* if for all $x, y, z, w \in V$,

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$,

$\mathcal{A}(V)$ is defined to be the set of all algebraic curvature tensors on V.

Definition 2. A *model space* (V, R) is a vector space paired with an algebraic curvature tensor on that space.

Definition 3. A *covariant tensor* on a vector space V is a tensor of type $(0, q)$, or a multilinear map from V^q to $\mathbb{R}$. Given an element U of V^q, or a $q-tuple$ of vectors, let U_i be the vector from the i^{th} space in the direct product.

The tensors of type (m, n) on a vector space V form another vector space, called $\mathfrak{T}^m_n$.

Definition 4. Given covariant tensors T_i on vector spaces V_i with $dim(V_i) \geq 1$, $V_i \subset W$, and $\bigcap_{i=1}^k V_i = \{0\}$, where W is a finite-dimensional vector space and $i \in \{1, 2......k\}$ for some $k \in \mathbb{N}$, define the *direct sum* $T = \bigoplus_{i=1}^k T_i$ on $V = \bigoplus_{i=1}^k V_i$ so that $T(U) := 0$ if $\exists\, i, j$ such that $U_i \in V_r$ and $U_j \in V_t$ for some $r \neq t,\ r, t \in \{1, 2, ...k\}$. In other words, $T(U) = 0$ if any two of the U_i come from different subspaces. If there exists some i such that $U_j \in V_i\ \forall\, i$,

$$T(U) := T_i(U)$$

Note. Since $V = \bigoplus_{i=1}^k V_i$, $v = \sum_{i=1}^k v_i$ for some unique $v_i \in V_i$. Therefore for a tuple $U \in V^q$, $U = \sum_{i=1}^k U^i$ for some unique $U^i \in V_i^q$. Then from the above definition one can see that $T(U) = \sum_{i=1}^k T(U^i)$.

Algebraic curvature tensors are covariant tensors of type $(0, 4)$, and if an algebraic curvature tensor can be written as a direct sum of other algebraic curvature tensors, it is said to be *decomposable*. Otherwise is is *indecomposable*.

Definition 5. For $R \in \mathcal{A}(V)$, the *kernel* of R is the subspace

$$ker(R) := \{v \mid R(v, x, y, z) = 0\, \forall\, x, y, z \in V\}.$$

Note. The identities in Definition 1 show that $\{v \mid R(v, x, y, z) = 0\, \forall\, x, y, z \in V\}$, $\{v \mid R(x, v, y, z) = 0\, \forall\, x, y, z \in V\}$, $\{v \mid R(x, y, v, z) = 0\, \forall\, x, y, z \in V\}$, and $\{v \mid R(x, y, z, v) = 0\, \forall\, x, y, z \in V\}$ are all the kernel of R.

Definition 6. For $R \in \mathcal{A}(V)$, $A^{\star}R(x, y, z, w) := R(Ax, Ay, Az, Aw)$. A is in the structure group G_R of R if $A^{\star}R = R$. It is assumed that $A \in GL_N(\mathbb{R})$.

Note. Strictly speaking, we should say $A \in G_R$ if A^{-1} leaves R fixed. However, this is equivalent to A leaving R fixed, so we will be concerned with A rather than A^{-1}.

Clarification 1. For the purposes of this paper, capital R will usually refer to an algebraic curvature tensor. V refers to a vector space. A usually refers to a square matrix, often an element of G_R. β refers to a basis for V and B_i a basis for V_i. Define (n) to be the set $\{1,\ 2,\ ....n\}$.

If $\beta = \{e_1,\ e_2,\ ....e_N\}$, then we can use column vectors to write coordinates for a given vector. For example,

$$e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where the one is in the i^{th} position. In general, a column vector v represents a linear combination that contains v_i copies of e_i for any $i\ \in\ [n]$.

Clarification 2. Let B be the matrix of a linear transformation defined on an N-dimensional vector space $V = \bigoplus_{i=1}^{k} V_i$. Let β_i be a basis for V_i. Let $\beta = \bigcup_{i=1}^{k} \beta_i$ be an ordered basis for V, with the elements of β_i listed before those of β_j if $i < j$. Let $\beta = \{e_1, e_2, ...e_N\}$. Say $p(i)$ is the smallest number such that $e_{p(i)} \in \beta_i$. Then define B^{ij} to be the $dim(V_i) \times dim(V_j)$ submatrix

$$B[p(i), p(i) + 1, ..., p(i + 1) - 1;\ p(j), p(j) + 1, ..., p(j + 1) - 1]$$

As a block matrix, B is written

$$B = \begin{bmatrix} B^{11} & \dots & B^{1k} \\ \vdots & \ddots & \vdots \\ B^{k1} & \dots & B^{kk} \end{bmatrix}$$

Let $[B^{ij}]$ be an inclusion of B^{ij} into $Gl(V)$ given by

$$B = \begin{bmatrix} 0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & 0 & \dots & \dots & \vdots \\ 0 & \dots & B^{ij} & \dots & \dots & \vdots \\ \vdots & \dots & 0 & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

or if we set $M = [B^{ij}]$, we have $M^{kl} = B^{ij}$ if $i = k$ and $j = l$, and $M^{kl} = 0$ otherwise.

3 Determination of G_R for decomposable R

In this section and the next we assume that in a direct sum of vector spaces, the basis used is the union of the bases of the subspaces in the direct sum ordered the same way as the indices on the spaces. This way we can use the notation set up in Clarification 2. The lemma below is adopted from [3].

Lemma 3.1. *If $R = \bigoplus_{i=1}^k R_i \in \mathcal{A}(V)$ has $ker(R) = \{0\}$, then $ker(R_i) = \{0\}$ for all $i \in (k)$.*

Lemma 3.2. *Suppose $(V, R) = \bigoplus_{i=1}^k (V_i, R_i)$ and $A \in G_R$. Suppose $x_i \in V$ for $i \in (1, 2, 3, 4)$. If there exist i, j such that $x_i = Av_1$ for some $v_1 \in V_k$ and $x_j = Av_2$ for some $v_2 \in V_l$ where $l \neq k$, then $R(x_1, x_2, x_3, x_4) = 0$.*

Proof. Without loss of generality, assume $x_1 = Av_1$ for some $v_1 \in V_k$ and $x_2 = Av_2$ for some $v_2 \in V_l$. Because $A \in GL_N(\mathbb{R})$ there exists an inverse element $A^{-1}v$ for all $v \in V$. Therefore $R(Av_1, Av_2, x_3, x_4) = A^*R(v_1, v_2, A^{-1}x_3, A^{-1}x_4) = R(v_1, v_2, A^{-1}x_3, A^{-1}y_4)$. But by the definition of $R = \bigoplus_{i=1}^k R_i$ (Definition 4) we have

$$R(v_1, v_2, A^{-1}x_3, A^{-1}x_4) = 0$$

because $v_1 \in V_l$, and $v_2 \in V_k$ with $l \neq k$.

□

Lemma 3.3. *Suppose $(V, R) = \bigoplus_{i=1}^k (V_i, R_i)$ with $V_i \neq \{0\}$ and (V_i, R_i) indecomposable. Also suppose $ker(R) = \{0\}$ and $A \in G_R$. Then for $i, j, t \in \{1, ..., k\}$, $A^{ij} \neq 0 \Leftrightarrow (A^{it} = 0 \ \forall t \neq j)$.*

Proof. Assume $A^{ij} \neq 0$. Set $B = \sum_{t \neq j} [A^{it}]$, so that $B^{uv} = A^{uv}$ if both $u = i$ and $v \neq j$, and $B^{uv} = 0$ otherwise. It suffices to show that $B = 0$. B is written

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \dots & \vdots & \dots & \vdots & \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ A^{i1} & \dots & A^{i(j-1)} & 0 & A^{i(j+1)} & \dots & A^{ik} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \dots & \dots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & \dots & 0 \end{bmatrix}$$

Since $A \in Gl(V)$, $rk(B + [A^{ij}]) = dim(V_i)$. $B + [A^{ij}]$ is the matrix containing the entire i^{th} block-row, with zeroes elsewhere. Assume the contrary; then $rk((B + [A^{ij}])^T) < dim(V_i)$. This means $(B + [A^{ij}])^T$ has a nontrivial kernel in V_i, but $(B + [A^{ij}])^T v = A^T v$ for $v \in V_i$, so A has a nontrivial kernel - a contradiction.

Because $rk(B + [A^{ij}]) = dim(V_i)$, $col(B + [A^{ij}]) = V_i$. Therefore $col(B) + col([A^{ij}]) = V_i$. Set $S = col(B) \cap col([A^{ij}])$. Find a basis β_s for S, extend β_s to bases $\beta_b \cup \beta_s$ for $col(B)$ and $\beta_s \cup \beta_a$ for $col([A^{ij}])$. Now

$$V_i = col(B) + col([A^{ij}]) = span\{\beta_a \cup \beta_s \cup \beta_b\}.$$

Therefore $V_i = span\{\beta_a\} \oplus span\{\beta_s \cup \beta_b\}$. Consider $R(x, y, z, w)$ where $x \in span\{\beta_a\}$, $y \in span\{\beta_s \cup \beta_b\}$, and $z, w \in V_i$. Since all four vectors are in V_i, $R(x, y, z, w) = R_i(x, y, z, w)$. Then $R_i(x, y, z, w) = R_i([A^{ij}]v_j, B\tilde{v}, z, w)$ for some $v_j \in V_j$ and $\tilde{v} \in \bigoplus_{r \neq j} V_r$. This is because $span\{\beta_a\} \subseteq col([A^{ij}])$ and $span\{\beta_s \cup \beta_b\} = col(B)$. The next calculation follows by the multilinearity of R:

$$\begin{aligned} R(Av_j, A\tilde{v}, z, w) &= R((A - [A^{ij}])v_j + [A^{ij}]v_j, (A - B)\tilde{v} + B\tilde{v}, z, w) \qquad (1) \\ &= R((A - [A^{ij}])v_j, (A - B)\tilde{v}, z, w) \\ &\quad + R([A^{ij}]v_j, (A - B)\tilde{v}, z, w) \\ &\quad + R((A - [A^{ij}])v_j, B\tilde{v}, z, w) \\ &\quad + R_i([A^{ij}]v_j, B\tilde{v}, z, w) \end{aligned}$$

It is easy to see, however, that $(A - [A^{ij}])v_j, (A - B)\tilde{v} \in \bigoplus_{r \neq i} V_r$. Because $(V, R) = (V_i, R_i) \oplus (\bigoplus_{r \neq i} V_r, \bigoplus_{r \neq i} R_r)$ and $z, w \in V_i$, one can see that all terms vanish but the last (see Definition 4). Therefore

$$R(Av_j, A\tilde{v}, z, w) = R_i([A^{ij}]v_j, B\tilde{v}, z, w) = R_i(x, y, z, w)$$

however, $R(Av_j, A\tilde{v}, z, w) = 0$ by Lemma 3.2.

It is not hard to see that the above proof technique does not depend on which one of x, y, z, or w is in $span\{\beta_a\}$ and which one is in $span\{\beta_s \cup \beta_b\}$. Since one of x, y, z, w being in $span\{\beta_a\}$ and another one being in $span\{\beta_s \cup \beta_b\}$ implies $R_i(x, y, z, w) = 0$, we have

$$(V_i, R_i) = (span\{\beta_a\}, \tilde{R}) \oplus (span\{\beta_s \cup \beta_b\}, \hat{R})$$

for some $\tilde{R}$ and $\hat{R}$. By assumption, R_i is indecomposable. Therefore either $span\{\beta_s \cup \beta_b\} = \{0\}$, or $span\{\beta_a\} = \{0\}$. If the former is true, then $col(B) = \{0\}$, so $B = 0$ as desired. If the latter is true, then $col([A^{ij}]) \subseteq col(B)$, in which case $V_i = col(B) = span\{\beta_s \cup \beta_b\} = span\{\beta_s\} \oplus span\{\beta_b\}$.

Assume $x \in span\{\beta_s\}$, $y \in span\{\beta_b\}$. So again, $x \in col([A^{ij}])$, $y \in col(B)$. Therefore $R_i(x, y, z, w) = R_i([A^{ij}]v_j, B\tilde{v}, z, w)$, and as was shown above in Equation 1, we have $R_i([A^{ij}]v_j, B\tilde{v}, z, w) = 0$. Again, $R(x, y, z, w) = 0$ regardless of which one of x, y, z, or w is chosen to be from $span\{\beta_s\}$ and which one is chosen to be from $span\{\beta_b\}$. Since one of x, y, z, w being in $span\{\beta_s\}$ and another one being in $span\{\beta_b\}$ implies $R_i(x, y, z, w) = 0$,

$$(V_i, R_i) = (span\{\beta_s\}, \tilde{R}) \oplus (span\{\beta_b\}, \hat{R})$$

for some $\tilde{R}$ and $\hat{R}$. Therefore either $span\{\beta_s\} = \{0\}$ or $span\{\beta_b\} = \{0\}$. The former cannot be true, for we have already assumed $span\{\beta_a\} = \{0\}$ and $col([A^{ij}]) = span\{\beta_a \cup \beta_s\}$. This would imply $[A^{ij}] = 0$, contradicting our hypotheses. Suppose the latter were true; then $V_i = col([A^{ij}] = col(B)$. Therefore $R_i(x, y, z, w) = 0$ for all x, y, z, w in V_i. By Lemma 3.1, $ker(R) \neq \{0\}$; a contradiction. Therefore it must be true that $span\{\beta_s \cup \beta_b\} = \{0\}$, so $B = 0$ as was to be shown. $\square$

Theorem 3.1. *Suppose* $(V, R) = \bigoplus_{i=1}^{k}(V_i, R_i)$ *with* $V_i \neq \{0\}$ *and* (V_i, R_i) *indecomposable. Also suppose* $ker(R) = \{0\}$ *and* $A \in G_R$*. Then there exists a permutation* σ *such that* $A : V_i \to V_{\sigma(i)}$.

Proof. Define n_i to be the number such that e_i is in the n_i^{th} basis of the β_i. Consider the function $f : (k) \to (k)$ such that $n_i \mapsto n_w$, where w is a number such that $a_{iw} \neq 0$, guaranteed to exist because $A \in GL_N(\mathbb{R})$. This function is well defined: Given a number $n \in K$, and any j such that $n_j = n$, suppose $a_{jw} \neq 0$ for some index w. Then $n_t = n_w$ for any other number t such that $a_{jt} \neq 0$. This follows from Lemma 3.3; if $a_{jw} \neq 0$ then $A^{n_j n_w} = A^{n n_w} \neq 0$, because it is a submatrix containing the entry a_{jw}. By Lemma 3.3, however, $A^{nt} = 0$ when $t \neq n_w$. So $a_{jt} = 0$ if $n_j = n$ and $n_t \neq n_w$.

Given an index $j \in (1, ..., N)$, there exists i such that $a_{ij} \neq 0$, because $A \in GL_N(\mathbb{R})$. Observe that $f : n_i \mapsto n_j$, so that f is onto. Since $f : (k) \to (k)$ is onto and (k) is a finite set, f must also be one to one. Therefore f is a permutation.

Consider $A\beta$, a new basis for V where $b_j = Ae_j = \sum a_{ij}e_i$ are the new basis vectors. It was shown, however, that f is one to one, so there is only one n

such that $A^{nn_j} \neq 0$, and $n = f^{-1}(n_j)$. We have $b_j = \sum a_{ij} e_i$, but $a_{ij} = 0$ if $n_i \neq n = f^{-1}(n_j)$. Therefore $b_j \in V_n$. Also, if $n_t = n_j$, then $b_t \in V_n$. So we can set $\sigma = f^{-1}$, and we have $n = f^{-1}(n_j) = \sigma(n_j)$. Now $A : V_j \to V_{\sigma(j)}$. □

Corollary 1. *Both $A : V_r \to V_t$ and the image of V_r under A being nontrivial is equivalent to having $A^{tr} \neq 0$, and is also equivalent to the condition $t = \sigma(r)$.*

Proof. The proof of this corollary follows from the proofs of Lemma 3.3 and the above theorem. □

4 Submatrices of $A \in G_R$

The structure group can either send a subspace to another one, or preserve it. To describe the structure group, it is necessary to understand when a subspace can be sent to another and what this would imply about elements that could do it. In this section, it will be assumed that (V, R) is as described in Theorem 3.1, that $A \in G_R$, and that σ is the permutation such that $A : V_i \to V_{\sigma(i)}$.

Lemma 4.1. *Suppose $A : V_r \to V_t$. Then $dim(V_r) = dim(V_t)$.*

Proof. If $A^{tr} \neq 0$, then $A : V_r \to V_t$. Then because A is one to one, we must have $dim(V_r) \leq dim(V_t)$. Suppose $dim(V_r) < dim(V_t)$. A^{tr} is a $dim(V_t) \times dim(V_r)$ matrix, so $(A^{tr})^T$ has a nontrivial kernel. This means $[A^{tr}]^T$ has a nontrivial kernel in V_t, but due to Lemma 3.3, $A^{ti} = 0$ if $i \neq r$. Therefore $[A^{tr}]^T v = A^T v$ for $v \in V_t$, so the kernel of $[A^{tr}]^T$ is a subset of the kernel of A^T. But A^T cannot have a nontrivial kernel, because $A \in GL_N(\mathbb{R})$, so this is a contradiction. □

Corollary 2. *A^{tr} has full rank.*

Proof. A^{tr} is a square matrix by Lemma , and if it had a nontrivial kernel, then A would have a nontrivial kernel in V_r.

□

Corollary 3. *If $A : V_r \to V_t$, then there exists an isomorphism $\Phi^{rt} : V_r \to V_t$ such that $e_{p(r)+i} \mapsto e_{p(t)+i}$ for $i \in \{0, ..., dim(V_r) - 1\}$. Let $[\Phi^{rt}]$ be the N by N matrix transformation representing the isomorphism Φ^{rt}; $[\Phi^{rt}]^{ij} = Id$ if $i = t$ and $j = r$, and $[\Phi^{rt}]^{ij} = 0$ otherwise.*

Definition 7. Define the inclusion $J_i : GL_{dim(V_i)}(\mathbb{R}) \hookrightarrow GL_N(\mathbb{R})$ as shown:

$$J_i(g) = \begin{bmatrix} Id & 0 & \dots & 0 \\ 0 & g & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & Id \end{bmatrix}.$$

An easy computation shows that $[\Phi^{rt}] J_r(A^{tr}) = [A^{tr}]$.

Lemma 4.2. *Suppose $A : V_r \to V_t$; then $[A^{tr}]^* R_t = R_r$.*

Proof. Consider $R(x, y, z, w)$ where $x, y, z, w \in V_r$.
Then $R(x, y, z, w) = R_r(x, y, z, w)$. Also, because $A \in G_R$, $R(Ax, Ay, Az, Aw) = R_r(x, y, z, w)$. But $A : V_r \to V_t$ implies $Ax, Ay, Az, Aw \in V_t$, so for all $x, y, z, w \in V_r$,

$$R_r(x, y, z, w) = A^* R(x, y, z, w) = R_t(Ax, Ay, Az, Aw) = A^* R_t(x, y, z, w).$$

But $x, y, z, w \in V_r$, so $Ax = [A^{tr}]x$, $Ay = [A^{tr}]y$, etc. Therefore

$$R_r(x, y, z, w) = A^* R_t(x, y, z, w) = [A^{tr}]^* R_t(x, y, z, w).$$

$\square$

Theorem 4.1. *Let $m(r) = dim(V_r)$. If $B \in GL_{m(r)}(\mathbb{R})$ and $J_r(B)^*([\Phi^{rt}]^* R_t) = R_r$, then the set $\{A^{tr} \mid A^{tr} \neq 0$ for some $A \in G_R\}$ is equal to the left coset BG_{R_r}.*

Proof. If $A^{tr} \neq 0$, then $A : V_r \to V_t$ and $\sigma(r) = t$. From Lemma 4.2, we know $[A^{tr}]^* R_t = R_r$. As noted above in observation 7, $[\Phi^{rt}] J_r(A^{tr}) = [A^{tr}]$, so $J_r(A^{tr})^*([\Phi^{rt}]^* R_t) = [\Phi^{rt}] J_r(A^{tr})^* R_t = [A^{tr}]^* R_t = R_r$ by Lemma 4.2. Note that both $[\Phi^{rt}]^* R_t$ and R_r are defined only on V_r. We can now restrict this discussion to V_r, so $(J_r(A^{tr})|_{V_r})^*([\Phi^{rt}]^* R_t) = R_r$. But $J_r(A^{tr})|_{V_r} = A^{tr}$, so $(A^{tr})^*([\Phi^{rt}]^* R_t) = R_r$. Also, B is an element of $GL_{m(r)}(\mathbb{R})$ so $B^*([\Phi^{rt}]^* R_t) = R_r$.

We wish to prove firstly that $A^{tr} \in BG_{R_r}$, and secondly that if $h \in BG_{R_r}$, then $h = A^{tr}$ for some $A \in G_R$. The first assertion follows because if $B^*([\Phi^{rt}]^* R_t) = R_r$ then $[\Phi^{rt}]^* R_t = (B^{-1})^* R_r$. So

$$\begin{aligned}(B^{-1} A^{tr})^* R_r &= (A^{tr})^*((B^{-1})^* R_r) \\ &= (A^{tr})^*([\Phi^{rt}]^* R_t) \\ &= R_r.\end{aligned}$$

Then $B^{-1} A^{tr} \in G_{R_r}$, which implies $A^{tr} \in BG_{R_r}$. The second assertion follows from the following construction: Let $A^{tr} = h$ and let $A^{rt} = h^{-1}$. Let $[h]$ denote $[A^{tr}]$ and $[h^{-1}]$ denote $[A^{rt}]$. If $i \notin \{r, t\}$, then let $A|_{V_i} = Id$. If $x, y, z, w \in \bigoplus_i V_i$ with $i \notin \{r, t\}$, then it is obvious that $A^* R(x, y, z, w) = R(x, y, z, w)$. Otherwise, one must consider the cases when $x, y, z, w \in V_r$ and when $x, y, z, w \in V_t$. If $x, y, z, w \in V_r$,

$$\begin{aligned}A^* R(x, y, z, w) &= [A^{tr}]^* R(x, y, z, w) \\ &= R_t([A^{tr}]x, [A^{tr}]y, [A^{tr}]z, [A^{tr}]w) \\ &= [h]^* R_t(x, y, z, w) \\ &= [\Phi^{rt}] J_r(h)^* R_t(x, y, z, w) \\ &= J_r(h)^*([\Phi^{rt}]^* R_t(x, y, z, w)),\end{aligned}$$

but $[\Phi^{rt}]^* R_t(x,y,z,w)$ is really only defined on V_r, so we can restrict to V_r and examine $h^*([\Phi^{rt}]^* R_t(x,y,z,w))$. We already know that $h = Bg$ for some $g \in G_{R_r}$. So

$$\begin{aligned} h^*([\Phi^{rt}]^* R_t(x,y,z,w)) &= (Bg)^*([\Phi^{rt}]^* R_t(x,y,z,w)) \\ = g^*(B^*([\Phi^{rt}]^* R_t(x,y,z,w))) &= g^* R_r(x,y,z,w) \\ &= R_r(x,y,z,w) \end{aligned}$$

so that $A^* R(x,y,z,w) = [h]^* R_t(x,y,z,w) = R_r(x,y,z,w) = R(x,y,z,w)$. We now consider the case $x,y,z,w \in V_t$. There exists an isomorphism $\Phi^{tr} : V_t \to V_r$ where $\Phi^{tr} = (\Phi^{rt})^{-1}$, and an easy computation shows $[\Phi^{tr}] J_t(A^{rt}) = [A^{rt}]$. So $[h^{-1}] = [\Phi^{tr}] J_t(h^{-1})$. In that case

$$\begin{aligned} A^* R(x,y,z,w) &= [A^{rt}]^* R(x,y,z,w) \\ &= R_r([A^{rt}]x, [A^{rt}]y, [A^{rt}]z, [A^{rt}]w) \\ &= [h^{-1}]^* R_r(x,y,z,w) \\ &= [\Phi^{tr}] J_t(h^{-1})^* R_r(x,y,z,w) \\ &= J_t(h^{-1})^*([\Phi^{tr}]^* R_r(x,y,z,w)). \end{aligned}$$

We already proved that $[h]^* R_t = J_r(h)^*([\Phi^{rt}]^* R_t) = R_r$ on V_r. We will use this to show that $J_t(h^{-1})^*([\Phi^{tr}] * R_r) = R_t$ fon V_t. Suppose $x,y,z,w \in V_r$; we now have

$$\begin{aligned} [h]^* R_t(x,y,z,w) &= R_r(x,y,z,w) \\ \Rightarrow [h^{-1}]^*([h]^* R_t(x',y',z',w') &= [h^{-1}]^* R_r(x',y',z',w') \\ \Rightarrow [h][h^{-1}]^* R_t(x',y',z',w') &= [h^{-1}]^* R_r(x',y',z',w') \end{aligned}$$

for any vectors $x',y',z',w' \in V_t$, because $[h^{-1}]$ maps x',y',z', and w' to vectors in V_r. Since $[h][h^{-1}]^* R_t$ and $[h^{-1}]^* R_r$ are only defined on V_t, they must be identical tensors. But by direct computation,

$$[h][h^{-1}] = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & Id & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix},$$

where $[h^{-1}][h]^{ij} = Id$ if $i = j = t$ and is zero otherwise. Then

$$\begin{aligned} [h][h^{-1}]^* R_t &= [\Phi^{tr}] J_t(h^{-1})^* R_r \\ \Rightarrow [h][h^{-1}]^* R_t &= J_t(h^{-1})^*([\Phi^{tr}]^* R_r) \end{aligned}$$

but because $x', y', z', w' \in V_t$, we have $[h][h^{-1}]^* R_t(x', y', z', w') = R_t(x', y', z', w')$ because $[h][h^{-1}] : V_t \to V_t$ and $[h][h^{-1}]|_{V_t} = Id$.
So $A^* R(x', y', z', w') = J_t(h^{-1})^*([\Phi^{tr}]^* R_r(x', y', z', w')) = R_t(x', y', z', w')$ for all $x', y', z', w' \in V_t$, completing the proof. $\square$

Note. The above theorem implies that given R, it is only possible to have $A : V_r \to V_t$ when R_r and $[\Phi]^* R_t$ are in the same orbit in $\mathcal{A}(V_r)$ under the action of $GL_{m(r)}(\mathbb{R})$.

5 Conclusions and Open Questions

The following conclusions characterize the structure group of a decomposable algebraic curvature tensor with trivial kernel in terms of the structure groups of its constituent tensors, if the relationships of those tensors are understood, i.e what matrix transformation turns one into the other and whether or not such a transformation exists. Future work will generalize the result to all covariant tensors of type $(0, n)$ with $n \geq 2$. It is easy to see that the symmetries of the curvature tensor were not needed in the methods used, and only three arguments are required.

1. If $A \in G_R$ and (V, R) is a decomposable model space with $ker(R) = \{0\}$, then A can only permute the subspaces into which V decomposes; it cannot send some vectors from one subspace to vectors with nonzero projections into multiple other subspaces.

2. When $(V, R) = \bigoplus_{i=1}^{k} (V_i, R_i)$, the set of possible nonzero block submatrices A^{tr} of $A \in G_R$ is equal to the coset BG_{R_r} where B is an endomorphism of V_r such that $B^*([\Phi^{rt}]^* R_t) = R_r$.

The open questions below are fairly ambitious, but their solution would result in a complete understanding of the structure group of any algebraic curvature tensor.

1. Determine the structure group of indecomposable algebraic curvature tensors.

2. Determine precisely when two indecomposable tensors $R_1, R_2 \in \mathcal{A}(V)$ are in the same orbit under $GL_N(\mathbb{R})$.

3. Given two arbitrary indecomposable algebraic curvature tensors R_1 and R_2 in the same orbit under $GL_N(\mathbb{R})$, figure out a way to find an element $B \in GL_N(\mathbb{R})$ with $B^* R_1 = R_2$. A constructive solution to the previous problem would solve this problem as well.

Acknowledgements

I am happy to thank C. Dunn for introducing me to the subject, and for being a great help throughout the program. Many thanks to R. Trapp for good conversations and interesting suggestions along the way. I am also very thankful to NSF grant DMS-0850959 and California State University, San Bernardino for funding this research.

References

[1] B. O'Niell, **Semi-Riemannian Geometry With Applications to Relativity**, Academic Press (1983), ISBN-13: 978-0-12-526740-3.

[2] P. Gilkey, **Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor**, World Scienific (2001), ISBN: 981-02-4752-4.

[3] C. Dunn, _______, J.Palmer. (2011) *On the Structure Group of Indecomposable Model Spaces.* arXiv:1108.2224v1 [math.GR]