Linear Independence of Sets of Three Algebraic Curvature Tensors

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Abstract

Looking at the dependence relationship between algebraic curvature tensors gives us an idea of how we can define an algebraic curvature tensor in terms of a linear combination of others. Given that φ is symmetric and ψ_i skew-symmetric, knowing when $\{R_\varphi, R_{\psi_1}, \ldots, R_{\psi_n}\}$ is linearly dependent is just one way of looking at this problem. In this study, we attempt to narrow down the problem by find some cases of when $\{R_\varphi, R_{\psi_1}, \ldots, R_{\psi_n}\}$ is linearly independent. We will show $\{R_\varphi, R_{\psi_1}\}$ and $\{R_\varphi, R_{\psi_1}, R_{\psi_2}\}$ are both linearly independent.

1 Introduction

Let V be a vector space with dimension n and let $x, y, z, w \in V$. Let V^* be the dual vector space. An *algebraic curvature tensor* $R \in \otimes^4(V^*)$ satisfies the following properties:

$$\begin{aligned} &R(x,y,z,w) = R(z,w,x,y) = -R(y,x,z,w), \text{and} \\ &R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0. \end{aligned} \tag{1}$$

The second is referred to as the *Bianchi Identity*. Let $\mathcal{A}(V)$ be the vector space of all algebraic curvature tensors on V.

Let φ and ψ be bilinear forms on V. We define φ to be *symmetric* if $\varphi(u,v) = \varphi(v,u)$ for all $u, v \in V$. We define ψ to be *skew-symmetric* if $\psi(u,v) = -\psi(v,u)$ for all $u, v \in V$. We define φ to be *positive definite* if $\varphi(v,v) > 0$ and is *negative definite* if $\varphi(v,v) < 0$ for all $v \in V$. If $\varphi(v,v) > 0$ for all $v \in V$, then v is called *spacelike*. If $\varphi(v,v) < 0$ for all $v \in V$, then v is called *timelike*.

Let $S^2(V^*)$ and $\Lambda^2(V^*)$ be the spaces of symmetric and skew-symmetric bilinear forms on V. If $\varphi \in S^2(V^*)$ and if $\psi \in \Lambda^2(V^*)$, then we define

$$\begin{aligned} R_\varphi(x,y,z,w) &= \varphi(x,w)\varphi(y,z) - \varphi(x,z)\varphi(y,w) \\ R_\psi(x,y,z,w) &= \psi(x,w)\psi(y,z) - \psi(x,z)\psi(y,w) - 2\psi(x,y)\psi(z,w). \end{aligned} \tag{2}$$

Note that $R_\varphi, R_\psi \in \mathcal{A}(V)$. In fact, [7]

$$span_{\psi \in \Lambda^2(V^*)}\{R_\psi\} = span_{\varphi \in S^2(V^*)}\{R_\varphi\} = \mathcal{A}(V).$$

If φ is positive definite and $\{e_1, \ldots, e_n\}$ is an orthonormal basis with respect to φ, then there exists a unique $\Theta : V \to V$ so that $\theta(x,y) = \varphi(\Theta x, y)$. Let A^* be the adjoint of A with respect to φ, characterized by the equation $\varphi(Ax, y) = \varphi(x, A^* y)$. If $A^* = A$, then we deine A to be *self-adjoint*. If $A^* = -A$, then we define A to be *skew-adjoint*.

Claim 1. *If $\theta \in S^2(V^*)$, then Θ is self-adjoint. If $\theta \in \Lambda^2(V^*)$, then Θ is skew-adjoint.*

Proof. If $\theta \in S^2(V^*)$, then

$$\varphi(\Theta x, y) = \theta(x,y) = \theta(y,x) = \varphi(\Theta y, x) = \varphi(x, \Theta y)$$

Since we know that $\varphi(Ax, y) = \varphi(x, A^* y)$, then $\Theta^* = \Theta$.

If $\theta \in \Lambda^2(V^*)$, then

$$\varphi(\Theta x, y) = \theta(x,y) = -\theta(y,x) = -\varphi(\Theta y, x) = \varphi((-\Theta)y, x) = \varphi(x, (-\Theta)y)$$

Since we know that $\varphi(Ax, y) = \varphi(x, A^* y)$, then $\Theta^* = -\Theta$. ∎

The form Θ can be represented by $n \times n$ matrix. If φ is positive definite and $\{e_1, \ldots, e_n\}$ is an orthonormal basis with respect to φ, then $\Theta_{ij} = \theta(e_j, e_i)$ and $[\Theta^*] = [\Theta]^T$, [4].

Furthermore, pending the same hypotheses, if $\theta \in S^2(V^*)$, then $[\Theta] = [\Theta]^T$ and if $\theta \in \Lambda^2(V^*)$ then, $-[\Theta] = [\Theta]^T$. So if $\theta \in S^2(V^*)$, then its matrix representation with respect to an orthonormal basis for φ is also symmetric and if $\theta \in \Lambda^2(V^*)$, then its matrix representation with respect to an orthonormal basis for φ is also skew-symmetric.

As we look at the relationship between R_φ, R_ψ, and R_τ, we will assume that φ is symmetric and positive definite, and that ψ and τ are skew-symmetric.

Claim 2. *If $A \in \otimes^2(V^*)$, then $\alpha R_A = \pm R_{\sqrt{|\alpha|}A}$.*

Proof. Suppose $A \in S^2(V^*)$.

$$\begin{aligned}
\alpha R_A(x, y, z, w) &= (\pm\sqrt{|\alpha|})^2[A(x, w)A(y, z) - A(x, z)A(y, w)] \\
&= (\pm\sqrt{|\alpha|})^2 A(x, w)A(y, z) - (\pm\sqrt{|\alpha|})^2 A(x, z)A(y, w) \\
&= \pm[A(\sqrt{|\alpha|}x, w)A(\sqrt{|\alpha|}y, z) - A(\sqrt{|\alpha|}x, z)A(\sqrt{|\alpha|}y, w)] \\
&= \pm R_{\sqrt{|\alpha|}A}
\end{aligned}$$

A similar proof will show the same for $\alpha R_A = \pm R_{\sqrt{|\alpha|}A}$ for $A \in \Lambda^2(V^*)$. □

Using this claim, we see that αR_A can be written as $\pm R_B$, where $B = \sqrt{|\alpha|}A$. So, our perspective on the problem shifts from studying what happens with $\sum_{i=1}^k \alpha_i R_{A_i} = 0$ to $\sum_{i=1}^k \epsilon_i R_{B_i} = 0$, where $\epsilon_i = \pm 1$. When studying the dependence relationship in a set of two canonical algebraic curvature tensors and when $\alpha_i \neq 0$, we can, then, refocus the study on the equation $R_{B_1} \pm R_{B_2} = 0$. Similarly, when studying the dependence relationship in a set of three algebraic curvature tensors and when $\alpha_i \neq 0$, we can study the equation $R_{B_1} + \epsilon R_{B_2} + \delta R_{B_2} = 0$, where $\epsilon, \delta = \pm 1$. We will focus most of our energy on studying the equation $R_\varphi + \epsilon R_\psi = \delta R_\tau$.

To help us study cases where $R_\phi = 0$, we provide the following definitions:

Definition 1. $\ker(\Phi) := \{v \in V \mid \Phi(v, w) = 0 \ \forall \ w \in V\}$.

Definition 2. $\ker(R_\Phi) := \{v \in V \mid R_\Phi(v, x, y, z) = 0 \ \forall \ x, y, z \in V\}$.

It is also important to note that if $\varphi \in S^2(V^*)$ and φ is non-degenerate or positive definite, then $\ker(\varphi) = \{0\}$. In addition to the given definitions, we will use the following result in [7] to give us the desired result for the proof in Theorem 4.2.

Lemma 1.1 (Gilkey, [7])**.** *If $\Phi \in S^2(V^*)$ or $\Phi \in \Lambda^2(V^*)$ and* $\text{Rank}\{\Phi\} \geq 2$*, then* $\ker(R_\Phi) = \ker(\Phi)$.

2 Motivation

To help understand the applications of this particular study for other areas in the realm of algebraic curvature tensors, we provide the following definitions:

Definition 3. $\eta(R) = inf\{k \mid R = \displaystyle\sum_{i=1}^k \alpha_i R_{\psi_i}$ where $\psi_i \in \Lambda^2(V^*)$ and $\alpha_i \in \mathbb{R}\}$.

Definition 4. $\eta(n) = sup\{\eta(R) \mid R \in \mathcal{A}(V)\}$.

Studying the results of linear dependence relationships of $\{R_\varphi, R_\psi, R_\tau\}$ allows us to narrow down the problem of finding an upper bound for $\eta(n)$. Specifically, we explore how many ψ_i's are needed, when given R_φ, so that $R_\varphi = \sum_{i=1}^{\eta(\varphi)} \epsilon_i R_{\psi_i}$, where $\epsilon_i = \pm 1$. We can conclude by the end of this paper that $\eta(\varphi) = 2$ is definitely not the upper bound over all such φ, in particular, those which are positive definite.

3 Previously Known Results

Before tackling the issue of whether or not $\{R_\varphi, R_\psi, R_\tau\}$ is linearly independent, we look at smaller cases involving the dependence relationship between R_φ and R_ψ. Utilizing Theorem 3.1 will help us specifically in studying $R_\varphi = -R_\psi$ as seen in Lemma 4.1.

Theorem 3.1. *Given $\varphi \in S^2(V^*)$, there exists a basis for V, $\{e_1^-, \dots, e_p^-, e_1^+, \dots, e_q^+, n_1, \dots, n_s\}$, so that*

1. $\varphi(e_i^\pm, e_j^\pm) = \pm\delta_{ij}$,
2. $\varphi(e_i^+, e_j^-) = 0$, *and*
3. $\varphi(n_i, x) = 0$ *for any* $x \in V$.

As we progress into studying $\{R_\varphi, R_\psi\}$ in Theorem 4.2, we look more closely at rank conditions. This requires us to use the following lemma. Theorem 3.3 will also give us the desired results for Theorem 4.2.

Lemma 3.2. (Gantmakher[5]). *If $\psi \in \Lambda^2(V^*)$, then Rank(ψ) must be even.*

Theorem 3.3. (Treadway[8]). *If $\psi \in \Lambda^2(V^*)$,* $\mathrm{Rank}(\psi) \geq 4$, *then there does not exist $\varphi \in S^2(V^*)$ so that $R_\varphi = R_\psi$.*

Studying the final set of three algebraic curvature tensors calls for assistance in certain areas. The following results will guide us through the proof for Theorem 4.3. As we study what happens when $R_\varphi + \epsilon R_\psi = \delta R_\tau$, it is also important to address the case where $R_\psi = \pm R_\tau$.

Lemma 3.4. (Gilkey[7]).

1. *Let $\Phi_i \in S^2(V^*)$. If Rank$(\Phi_1) \geq 3$ and if $R_{\Phi_1} = R_{\Phi_2}$, then $\Phi_1 = \pm\Phi_2$.*
2. *Let $\Phi_i \in \Lambda^2(V^*)$. If $R_{\Phi_1} = R_{\Phi_2}$, then $\Phi_1 = \pm\Phi_2$.*

Theorem 3.5 (Treadway [8]). *If $\psi \in \Lambda^2(V^*)$ and Rank$(\psi) = 2k \geq 4$, then there does not exist $\tau \in \Lambda^2(V^*)$ such that $R_\psi + R_\tau = 0$.*

4 Results

Lemma 4.1. *If $\psi \in \Lambda^2(V^*)$,* $\mathrm{Rank}(\psi) \geq 4$, *there does not exists $\varphi \in S^2(V^*)$ so that $R_\varphi = -R_\psi$.*

Proof. For the sake of contradiction, suppose that there exists $\varphi \in S^2(V^*)$ so that $R_\varphi = -R_\psi$.

The proposed solution φ may be negative definite with signature $(p, 0)$, positive definite with signature $(0, q)$, or signature (p, q) where $p, q \neq 0$. There, then, exists an orthonormal basis $\{e_1^-, \dots, e_p^-, e_1^+, \dots, e_q^+, n_1, \dots, n_s\}$ with respect to φ by Lemma 3.1. Since it is implied that $\mathrm{Rank}(\varphi) \geq 4$, there exists at least two spacelike vectors $\{e_1^+, e_2^+\}$ or at least two timelike vectors $\{e_1^-, e_2^-\}$. In the case that φ has at least two spacelike vectors, consider $\{e_1^+, e_2^+\}$. Since $\{e_1^+, e_2^+\}$ are orthonormal, we get

$$\begin{aligned} R_\varphi(e_1^+, e_2^+, e_2^+, e_1^+) &= \varphi(e_1^+, e_1^+)\varphi(e_2^+, e_2^+) - \varphi(e_1^+, e_2^+)\varphi(e_2^+, e_1^+) \\ &= 1 \cdot 1 - 0 \\ &= 1 \end{aligned}$$

And,

$$\begin{aligned} -R_\psi(e_1^+, e_2^+, e_2^+, e_1^+) &= -[\psi(e_1^+, e_1^+)\psi(e_2^+, e_2^+) - \psi(e_1^+, e_2^+)\psi(e_2^+, e_1^+) - 2\psi(e_1^+, e_2^+)\psi(e_2^+, e_1^+)] \\ &= -[0 + 3\psi(e_1^+, e_2^+)^2] \\ &= -3\psi(e_1^+, e_2^+)^2 \end{aligned}$$

Which is to say

$$1 = R_\varphi(e_1^+, e_2^+, e_2^+, e_1^+) = -R_\psi(e_1^+, e_2^+, e_2^+, e_1^+) = -3\psi(e_1^+, e_2^+)^2$$

This implies that $-\dfrac{1}{3} \geq 0$. We arrive at a contradiction. We arrive at a similar result when we study the case where φ has at least two timelike vectors and we consider $\{e_1^-, e_2^-\}$ being orthonormal.

□

Using Lemma 4.1 will help us with our penultimate result, Theorem 4.2 as we study the relationship between R_φ and R_ψ as we near the final result.

Theorem 4.2. *If $\psi \in \Lambda^2(V^*)$, $\mathrm{Rank}(\psi) \geq 4$, and $\varphi \in S^2(V^*)$ be a positive definite inner product, then $\{R_\varphi, R_\psi\}$ is linearly independent.*

Proof. **Case 1a:** Suppose $R_\varphi = 0$. If $R_\varphi = 0$ then $\ker(R_\varphi)$ is equal to the whole vector space which, according to Lemma 1.1, must also equal $\ker(\varphi)$. However, φ is a positive definite inner product and has full rank, which is to say that $\ker(\varphi) = 0$. This provides a contradiction.

Case 1b: Suppose $R_\psi = 0$. We know from Lemma 1.1 that $\ker(R_\psi) = \ker(\psi)$ if $\mathrm{Rank}(\psi) \geq 2$. Then if $R_\psi = 0$, $\ker(R_\psi)$ must be all of the vector space. However, if that is true then $\mathrm{Rank}(\psi) \ngeq 2$. Therefore $\mathrm{Rank}(\psi) \leq 1$. Since $\psi \in \Lambda^2(V^*)$ and the rank of ψ must be even according to Theorem 3.2, then $\mathrm{Rank}(\psi) = 0$. However, as one of the conditions in the theorem, $\mathrm{Rank}(\psi) \geq 4$. This provides a contradiction.

Case 2: Suppose $R_\varphi = \pm R_\psi$. According to Theorem 3.3, however, there does not exist $\psi \in \Lambda^2(V^*)$ so that $R_\varphi = R_\psi$. Also according to Lemma 4.1, there does not exist $\psi \in \Lambda^2(V^*)$ so that $R_\varphi = -R_\psi$. We arrive at a contradiction.

□

We now reach the culmination of all our work. Using the previous results attained with Lemma 4.1 and Theorem 4.2, we can now satisfy our curiosity as to what happens with $\{R_\varphi, R_\psi, R_\tau\}$.

Theorem 4.3. *Let $\varphi \in S^2(V^*)$ be a positive definite inner product. Let $\psi, \tau \in \Lambda^2(V^*)$, $\mathrm{Rank}(\psi) \geq 4$, $\mathrm{Rank}(\tau) \geq 4$, and $\psi \neq \lambda\tau$ for any $\lambda \in \mathbb{R}$. Then $\{R_\varphi, R_\psi, R_\tau\}$ is linearly independent.*

Proof. Theorem 4.2 illustrates that $R_\varphi = 0$, $R_\psi = 0$, $R_\tau = 0$, and $R_\varphi = \pm R_\psi$ all provide contradictions. We are left only to deal with two more cases:

$$\begin{aligned} &(1)\quad R_\psi = \pm R_\tau \\ &(2)\quad R_\varphi + \epsilon R_\psi = \delta R_\tau \end{aligned}$$

Case 1: Suppose $R_\psi = \pm R_\tau$. Lemma 3.5 has shown that if $\mathrm{Rank}(\tau) \geq 4$ then there does not exist τ so that $R_\psi = -R_\tau$. Theorem 3.4 has also shown that if $R_\psi = R_\tau$ then $\psi = \pm\tau$, which is to say that $\tilde{\psi} = \lambda\tilde{\tau}$ since we have used the property that $\alpha R_\varphi = R_{\sqrt{|\alpha|}\varphi}$ for some $\alpha \in \mathbb{R}$ to change from $aR_{\tilde{\psi}} = bR_{\tilde{\tau}}$ to $R_\psi = \pm R_\tau$. However, we assumed that $\psi \neq \lambda\tau$, which provides a contradiction.

Case 2: Suppose $R_\varphi = \epsilon R_\psi + \delta R_\tau$ and also suppose s, t, p, q are distinct indices so that when we compute $R_\varphi(e_s, e_t, e_t, e_s) + \epsilon R_\psi(e_s, e_t, e_t, e_s) = \delta R_\tau(e_s, e_t, e_t, e_s)$ we are given this result

$$\tau_{ts}^2 = \frac{1 + 3\epsilon\psi_{ts}^2}{3\delta}. \tag{3}$$

And when we compute $R_\varphi(e_s, e_t, e_p, e_s) + \epsilon R_\psi(e_s, e_t, e_p, e_s) = \delta R_\tau(e_s, e_t, e_p, e_s)$ we are given this result

$$\epsilon\psi_{ts}\psi_{sp} = \delta\tau_{ts}\tau_{sp}. \tag{4}$$

Square both sides of Equation (2) and plug in $\tau_{ts}^2 = \dfrac{1 + 3\epsilon\psi_{ts}^2}{3\delta}$ and $\tau_{sp}^2 = \dfrac{1 + 3\epsilon\psi_{sp}^2}{3\delta}$ to achieve this equation

$$\psi_{ts}^2 + \psi_{sp}^2 = -\frac{\epsilon}{3}. \tag{5}$$

Doing the same process using $\{e_s, e_t, e_q, e_s\}$ and $\{e_s, e_p, e_q, e_s\}$ will yield, respectively

$$\psi_{ts}^2 + \psi_{sq}^2 = -\frac{\epsilon}{3} \tag{6}$$

and

$$\psi_{sp}^2 + \psi_{sq}^2 = -\frac{\epsilon}{3} \tag{7}$$

It becomes evident that $\psi_{sp}^2 = \psi_{sq}^2$ from Equations (3) and (4). Using this result for Equation (5), we can see that $\psi_{sp}^2 = -\frac{\epsilon}{6}$. This is to say that every non-diagonal entry in ψ is nonzero.

Replacing t with q in Equation (2) and multiplying both sides by τ_{ts}, we get

$$\epsilon\psi_{sp}\psi_{sq}\tau_{ts} = \delta\tau_{sp}\tau_{sq}\tau_{ts}. \tag{8}$$

Plug in $\delta\tau_{sq}\tau_{ts} = \epsilon\psi_{ts}\psi_{sq}$ to get

$$\psi_{sq}[\psi_{sp}\tau_{ts} - \tau_{sp}\psi_{ts}] = 0. \tag{9}$$

We know that $\psi_{sq} \neq 0$ so then

$$\tau_{ts} = \frac{\tau_{sp}\psi_{ts}}{\psi_{sp}}. \tag{10}$$

Plug in Equation (8) into Equation (2) to get

$$\tau_{sp}^2 = \frac{\epsilon\psi_{sp}^2}{\delta}. \tag{11}$$

However, we know that $\tau_{sp}^2 = \frac{1 + 3\epsilon\psi_{sp}^2}{3\delta}$, which is to say that $1 + 3\epsilon\psi_{sp}^2 = 3\epsilon\psi_{sp}^2$. Therefore we have a contradiction.

□

5 Open Questions

1. Is $\{R_\varphi, R_\psi, R_\tau\}$ still linearly independent when $\text{Rank}(\psi_i) \geq 2$?
2. Is $\{R_\varphi, R_{\psi_1}, R_{\psi_2}, R_{\psi_3}\}$, where φ is a symmetric bilinear form and ψ_i are skew-symmetric bilinear forms, linearly dependent?
3. When is it the case that $\{R_\varphi, R_{\psi_1}, R_{\psi_2}, \ldots, R_{\psi_n}\}$ is linearly dependent and ψ_i's form a Clifford family?

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