Structure Groups of Algebraic Curvature Tensors in Dimension Three

Lea Beneish
Research Experience for Undergraduates
California State University, San Bernardino
August 22, 2013

Abstract

The purpose of this paper is to examine structure groups of algebraic curvature tensors in dimension three. In dimension two, the structure group of algebraic curvature tensors is known to always be $SL_2^{\pm}$. In dimension three, there is more than one non-zero algebraic curvature tensor, up to multiple, which complicates the task. The two main goals used in this paper were, first, to consider an invertible 3×3 matrix A, one of each Jordan Normal form, and find the subspace of algebraic curvature tensors preserved by A. In particular, in the cases where the subspace of curvature tensors preserved by a given matrix was of dimension one, the algebraic curvature tensor that spans that subspace is found. This is to see, given an element A, is it in the structure group of some algebraic curvature tensor.The second goal is to classify the Lie algebras of the three non-trivial cases of Lie algebras of the structure groups of algebraic curvature tensors in dimension three which are known. The idea is to categorize these Lie algebras and then given a Lie group, it is possible to see if it is a structure group of some algebraic curvature tensor based on certain properties of its Lie algebra. The general question is: given a subgroup H of GL_n, does there exist an algebraic curvature tensor R such that its structure group is H?

1 Introduction

The structure group of an algebraic curvature tensor is the group of matrices that fix the tensor, in other words, its group of symmetries. The general problem is to explain the relationship between an algebraic curvature tensor R and its structure group. In particular, given a subgroup of GL_n, does there exist an algebraic curvature tensor R for which that subgroup is a structure group of R? Kaylor showed [6] that the structure group of an algebraic curvature tensor on a vector space of dimension two is $SL_2^{\pm}$, which is the group of endomorphisms of V with determinant plus or minus one. The case with the algebraic curvature tensor in three dimensions is more complicated because, in the previous case, there was only one algebraic curvature tensor entry up to the symmetries given in Definition 1, and this time, there are six. In order to examine this question in dimension three, Kaylor posed the opposite question and developed a method to consider a single matrix A in a subgroup of GL_n, the group of invertible $n \times n$ matrices, and ask if there is an algebraic curvature tensor such that R is preserved by A. The matrices considered were the various Jordan Normal forms for 3×3 matrices because every matrix is conjugate to a unique matrix in Jordan Normal form (up to rearrangements of Jordan Blocks) [3]. This method finds the dimension of the subspace of algebraic curvature tensors preserved by A. The second method used to examine this question was to study a classification of Lie algebras in order to see, given a Lie group, if it is the structure group of an algebraic curvature tensor. Obeidin showed [8] that all of the nontrivial structure groups of algebraic curvature tensors in three dimensions fall into one of three cases: Lie group dimension three, four, or six. Classifying these Lie algebras can make it possible to see, given a Lie group, if it is a structure group of some algebraic curvature tensor.

2 Preliminaries

The following includes definitions and motivating examples for the method and results in this paper.

2.1 Algebraic Curvature Tensors

Definition 1. *Let V be a vector space and $R : V \times V \times V \times V \to \mathbb{R}$ be linear in each slot. R is an algebraic curvature tensor if the following properties hold $\forall x, y, z, w \in V$:*

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$.

The vector space of algebraic curvature tensors is denoted $\mathcal{A}(V)$.

Claim 1. $R(x, x, y, z) = 0$.

Proof. $R(x, x, y, z)$ must satisfy the first property in definition 1, so that $R(x, x, y, z) = -R(x, x, y, z)$ but $R(x, x, y, z) = -R(x, x, y, z) \implies 2R(x, x, y, z) = 0$, so $R(x, x, y, z) = 0$. □

Definition 2. *Let $x, y, z, w \in V$ and let φ be symmetric ($\varphi(x, y) = \varphi(y, x)$) and bilinear.*
The canonical algebraic curvature tensor, R_φ is defined as

$$R\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

It is not difficult to check, using the properties in Definition 1, that R_φ is in fact an algebraic curvature tensor. [3]

2.2 Structure Groups

The following includes a definition of the structure group of an algebraic curvature tensor, two simple examples of structure groups to illustrate, and a proof of the fact that the structure group is in fact a group.

$R(Ax, Ay, Az, Aw)$ is the algebraic curvature tensor $R(x, y, z, w)$ precomposed with A, and is denoted $A^*R(x, y, z, w)$.

Definition 3. *Given $R \in A(v)$, the structure group of R is $\{A \in GL(V)|R(x, y, z, w) = R(Ax, Ay, Az, Aw)\}$ and is denoted G_R.*

Below are two examples of structure groups of algebraic curvature tensors:

Example 1. *When R is the zero tensor, the structure group G_R is $GL_n(V)$.*
This is true because If $R(x, y, z, w) = 0$ then A can be any matrix in $GL_n(V)$ and $0 = R = A^*R = 0$ will always hold.

Lemma 1. *[6] Let V be of dimension two, $A \in GL_2(V)$, and $R \in \mathcal{A}(V)$, then $A^*R = (detA)^2 R$.*

Proof. $R(e_1, e_2, e_2, e_1)$ this is the only nonzero entry of R in dimension two up to the symmetries listed in Definition 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $detA = ad - bc$. Then $Ae_1 = ae_1 + be_2$ and $Ae_2 = ce_1 + de_2$ and so

$$\begin{aligned}
& A^*R(e_1, e_2, e_2, e_1) = R(Ae_1, Ae_2, Ae_2, Ae_1) \\
= \;& R(ae_1 + be_2, ce_1 + de_2, ce_1 + de_2, ae_1 + be_2) \\
= \;& R(ae_1, de_2, ce_1, be_2) + R(ae_1, de_2, de_2, ae_1) + R(be_2, ce_1, de_2, ae_1) + R(be_2, ce_1, ce_1, be_2) \\
= \;& abcdR(e_1, e_2, e_1, e_2) + a^2d^2R(e_1, e_2, e_2, e_1) + abcdR(e_2, e_1, e_2, e_1) + b^2c^2R(e_2, e_1, e_1, e_2) \\
= \;& (ad - bc)^2R(e_1, e_2, e_2, e_1).
\end{aligned}$$

□

Example 2. *If V is of dimension two, $0 \neq R \in \mathcal{A}(V)$, and $A \in G_R$, then by the previous lemma $A^*R = (detA)^2R \Leftrightarrow detA = \pm 1$ and this group (the structure group of this R) is called $SL_2(V)^\pm$.*

Claim 2. *The structure group G_R is in fact a group:*

Proof. Let $A, B \in G_R$, then closure, associativity, identity, and inverses are checked, in that order:

1. $(AB)^*R(x,y,z,w) = R(ABx, ABy, ABz, ABw) = A^*R(Bx, By, Bz, Bw)$
   $= A^*(B^*R(x,y,z,w))$
   $= A^*R(x,y,z,w)$ since B is in the structure group of R
   $= R(x,y,z,w)$ since A is in the structure group of R
   Thus, $AB \in G_R$.

2. Since the operation being considered is composition of functions, this set is known to be associative.

3. The structure group contains the identity element because $I^*R(x,y,z,w) = R(x,y,z,w)$ since $Ix = x$, $Iy = y, Iz = z, and\ Iw = w$.

4. By assumption, if an element A is in the structure group of R, then A^{-1} exists. Since A is in the structure group $A^*R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w) = R(A^{-1}x, A^{-1}y, A^{-1}z, A^{-1}w)$, but this implies that $R(x,y,z,w) = (A^{-1})^*R(x,y,z,w)$ thus $A^{-1} \in G_R$.

$\square$

2.3 Jordan Normal Form

Since part of what is done to examine the relationship between algebraic curvature tensors and their structure groups is to look at what dimension the subspaces of algebraic curvature tensors are preserved by matrices A, one of each Jordan Normal form, the following includes a definition of Jordan Normal forms, the theorem that motivated this method (the fact that every matrix is conjugate to a matrix in Jordan Normal form), and a list of the four Jordan Normal forms examined in dimension three.

Definition 4. *A real Jordan Block of size k is the $k \times k$ matrix composed of eigenvalues $\lambda \in \mathbb{R}$, denoted $J(k,\lambda)$, is defined as*

$$J(k,\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}.$$

A Jordan Block of size k corresponding to the complex conjugate eigenvalues $a \pm bi$ is the $2k \times 2k$ matrix denoted $J(k,a,b)$, is defined as

$$J(k,a,b) = \begin{pmatrix} A & I & 0 & \dots & 0 & 0 \\ 0 & A & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & A & I \\ 0 & 0 & 0 & \dots & 0 & A \end{pmatrix}, \text{ where } A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, b > 0, \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose A_i are square matrices $i = 1 \dots n$, define $\bigoplus_{i=1}^n A_i = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_n \end{pmatrix}$

Matrices in which the A_i are Jordan Blocks are called Jordan Matrices; these matrices are said to be in Jordan Normal Form.

Theorem 1. *[3] For all $A : V \to V$, there exists a unique Jordan Normal Form of A up to rearrangement of blocks. In other words, every matrix is conjugate to a unique matrix in Jordan Normal Form.*

In examining the question of structure groups of algebraic curvature tensors on a vector space of dimension three, it is useful to look at one matrix for every possible Jordan Normal Form for matrices of dimension three (there are four, listed below), because every 3×3 matrix is conjugate to one of these forms, so considering each form gives information about all possible structure groups containing a given element.

It is easy to check that the Jordan Normal Forms for three dimensional matrices with real entries are as follows:
1. $J(3, \lambda)$,
2. $J(2, \lambda) \oplus J(1, \eta)$,
3. $J(1, \lambda) \oplus J(1, \eta) \oplus J(1, \tau)$,
4. $J(1, \lambda) \oplus J(2, a, b)$.

2.4 Lie Groups and Lie Algebras

The structure groups of these algebraic curvature tensors are Lie groups (see Definiton 5). The second method of looking at the problem of finding the structure group for algebraic curvature tensors in dimension three involved looking at Obeidin's [8] cases of the Lie algebras (see Definition 6) of the structure groups in dimension three and classifying them. The idea being that given a Lie group, the classification of these Lie algebras could determine if it is the structure group of some algebraic curvature tensor.

Definition 5. *A Lie Group is a group that is also a differentiable manifold such that the group operations $(g_1, g_2) \mapsto g_1 g_2$ and $g \mapsto g^{-1}$ are smooth.*

The following examples of Lie Groups are well known:

Example 3. $GL_n(\mathbb{R})$=$\{A \in M_n(\mathbb{R}) | det(A) \neq 0\}$.

Example 4. $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | det(A) = 1\}$.

Example 5. $SL_n^{\pm}(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | det(A) = \pm 1\}$.

Example 6. $O(n)$=$\{A \in M_n(\mathbb{R}) | A^T A = I\}$.

Example 7. $SO(n) = O(n) \cap Sl(n)$.

More examples can be found in [4].

Definition 6. *A Lie algebra is a vector space over a field of characteristic zero (for the purposes of this project, use $F = \mathbb{R}$) with a bilinear form [a, b] that is skew symmetric $[a, b] = -[b, a]$ and satisfies the Jacobi Identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.*

Example 8. $\mathfrak{gl}(n)$ *is the Lie algebra of $n \times n$ matrices.*

Let V be a vector space over a field of characteristic zero. The set of linear transformations from $V \to V$ is a ring with the operation of multiplication, but with the Lie bracket operation, it becomes a Lie algebra. When the Lie algebra is embedded into a matrix ring, the Lie bracket is defined as $[a, b] = ab - ba$, it is a skew-symmetric bilinear form satisfying the Jacobi identity.

The Lie algebra associated to the Lie group is the tangent space of the manifold associated to the identity matrix I. The exponential map exp for matrices is defined as $exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ and $A^0 = I$; this maps elements from the Lie algebra to the Lie group, $exp : \mathfrak{g} \to G$ [5].

Below is a short demonstration of this relationship between Lie group and Lie algebra.

Example 9. *$\mathfrak{o}(n)$ is the Lie algebra consisting of skew-symmetric matrices s.t. $A^T = -A$.*

Take the Lie group $O(n) = \{AA^T = I\}$, A_t is a path in $O(n)$ and $A_0 = I$. Then $\frac{d}{dt}(A_t)|_{t=0}$ is in the Lie algebra of $O(n)$ $\frac{d}{dt}(A_tA_t^T)|_{t=0} = \frac{d}{dt}|_{t=0}(I) = 0$ but $\frac{d}{dt}(A_tA_t^T)|_{t=0} = \frac{d}{dt}(A_t)|_{t=0} * A_t^T|_{t=0} + A_t|_{t=0} * \frac{d}{dt}A_t^T|_{t=0}$=$A(0)I + IA^T(0) = A(0) + A^T(0) = 0$ so $\mathfrak{o}(n)$ consists of matrices such that $A^T = -A$ (i.e. the skew symmetric matrices).

Below are a few properties of Lie algebras, used later in categorization of Lie algebras of known structure groups:

Definition 7. *The structure constants of a Lie algebra are defined as C_{ijk} for $[X_i, X_j] = \sum_{i=1}^{n} C_{ijk}X_k$.*

Definition 8. *A Lie algebra isomorphism is a bijective map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that:*
1. ϕ is a linear map, $\phi(cX + dY) = c\phi(X) + d\phi(Y)$ $\forall c, d \in \mathbb{R}$, and
2. $\phi([X, Y])_{\mathfrak{g}_1} = [\phi(X), \phi(Y)]_{\mathfrak{g}_2}$.

Proposition 2. *[1] Let $\mathfrak{g}_1$ $\mathfrak{g}_2$ be two Lie algebras of finite dimension. Suppose each has a basis with respect to which the structure constants are the same. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*

Definition 9. *A Lie algebra is said to be abelian if $[a, b] = 0$ for all $a, b \in \mathfrak{g}$.*

Definition 10. *If a subspace $I \subseteq \mathfrak{g}$ satisfies the property that $[I, \mathfrak{g}] \subseteq I$, then I is an ideal in the Lie algebra $\mathfrak{g}$.*

Definition 11. *A Lie algebra is said to be simple if it is non-abelian and its only ideals are 0 and itself.*

Definition 12. *A Lie algebra is said to be semisimple if it is a direct sum of simple Lie algebras.*

Definition 13. *The adjoint map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by $ad_{X'}(Y') = [X', Y']$ for $X', Y' \in \mathfrak{g}$. [1]*

Theorem 3. *[4] An equivalent condition for a Lie algebra to be semisimple is for its Killing form $k(x, y) = tr(ad(x)ad(y))$ is non-degenerate.*

Obeidin found all cases for possible Lie algebras of the structure groups for algebraic curvature tensors in dimension three and part of this project was to find the Lie bracket relations in order to classify these Lie algebras and determine whether or not they are semisimple with the goal of being able to, given a Lie group, see if it the structure group of some algebraic curvature tensor by looking at its Lie algebra.

3 Method

3.1 Jordan Normal Forms

Let V be a vector space and let $\{e_i\}$ be a basis for V. The $R(e_i, e_j, e_k, e_l)$ can be enumerated as R_1, R_2, etc. In the case where V is of dimension three, there are six possible non-zero algebraic curvature entries on any basis $\{e_1, e_2, e_3\}$, up to these symmetries and we define the R_i as follows:

$$\begin{aligned}
R_1 &= R(e_1, e_2, e_2, e_1), \\
R_2 &= R(e_1, e_3, e_3, e_1), \\
R_3 &= R(e_2, e_3, e_3, e_2), \\
R_4 &= R(e_1, e_2, e_3, e_1), \\
R_5 &= R(e_2, e_1, e_3, e_2), \\
R_6 &= R(e_3, e_1, e_2, e_3).
\end{aligned}$$

From each Jordan Normal form, one can find relations for the algebraic curvature tensors by writing each entry of $R(Ae_i, Ae_j, Ae_k, Ae_l)$, as linear combinations according to the values in the matrix A. Then, this can be expanded to get a collection of relations that must hold (see, for example, Section 4.1).

Given a matrix $A \in GL_n$, the relations $A^*R(e_i, e_j, e_k, e_l) = R(e_i, e_j, e_k, e_l)$ can be expressed as $Kx = x$ for some matrix K (Kaylor [6] was the first to study this matrix), where $x = [R_1, R_2, ..., R_6]^T$. This is equivalent to $(K - I)x = 0$ which has solutions when $x = 0$ (which is the trivial case of the zero tensor) or when $det(K - I) = 0$,which has dimension equal to the nullity of $K - I$. So the solution space, the set of algebraic curvature tensors that are preserved by the matrix A, has the same dimension as the nullity of $K - I$.

A lattice is constructed by computing the nullities of $K - I$ for the conditions when each element of the complete factorization is equal to zero; this forms the base level or first row of the lattice. Higher levels are constructed by choosing any two elements of the first row and finding the nullity of $K - I$ under the condition that they are both satisfied, and subsequently checking of those conditions simultaneously satisfy any other elements on the base level. This process is repeated for higher levels of the lattice until all entries in A are determined.

In particular, the cases of nullity one are of interest because if it is possible to find an algebraic curvature tensor that is preserved by the matrix A, then that curvature tensor spans the space of algebraic curvature tensors that are preserved by that matrix.

In sum, there are five main steps to this process:

1. Find the relations $A^*R(e_i, e_j, e_k, e_l) = R(e_i, e_j, e_k, e_l)$ and express them in a matrix K.
2. Compute the determinant of $K - I$ (factor, if needed).
3. Find solutions to $det(K - I) = 0$ compute the nullity of $K - I$ when those factors are equal to zero.
4. Display this information in a lattice by putting irreducible factors of the characteristic polynomial on the lowest level and connecting dots on the higher levels of the lattice to indicate a situation in which multiple factors equal zero.
5. In cases of nullity one, find the algebraic curvature tensor that spans the solution space.

3.1.1 Previous results

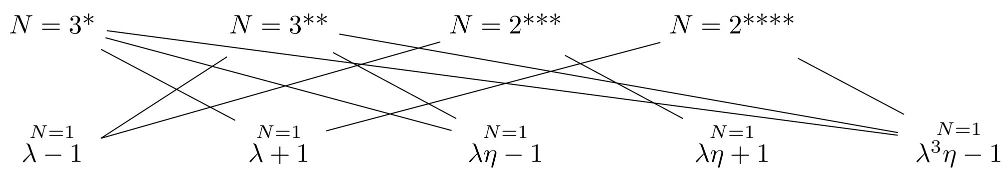
The results in this (sub)subsection are due to Kaylor [6]

As before, the four cases for the Jordan Normal forms of dimension three with real entries are $J(3, \lambda)$, $J(2, \lambda) \oplus J(1, \eta)$, $J(1, \lambda) \oplus J(1, \eta) \oplus J(1, \tau)$, and $J(2, a, b) \oplus J(1, \lambda)$. The case with three distinct real eigenvalues and the case with two complex conjugate eigenvalues are part of the results section (i.e. new). The case with one eigenvalue of multiplicity three and the case with one eigenvalue of multiplicity two and one other real eigenvalue have been analyzed previously. The results are as follows:

In the case with one eigenvalue of multiplicity three, the determinant of $K - I$ is $(\lambda^4 - 1)^6$. It is clear that the determinant equals zero iff $\lambda = \pm 1$. When $\lambda = \pm 1$, the rank of $K - I$ is four and thus the nullity is two. So, the space of algebraic curvature tensors preserved by this matrix A is two dimensional. The lattice is as follows:

$$\overset{N=2}{\lambda + 1} \qquad\qquad \overset{N=2}{\lambda - 1}$$

In the case with one eigenvalue of multiplicity two and one different real eigenvalue, the determinant of $K-I$ is $(\lambda^4-1)(\lambda^2\eta^2-1)^3(\lambda^3\eta-1)^2$. This determinant is equal to zero when $\lambda^4-1=0$, when $\lambda^2\eta^2-1=0$, and when $\lambda^3\eta-1=0$ (or a combination of these). Kaylor found that for each of these individual factors to be equal to zero, the nullity of the matrix $K-I$ is 1. In the case that both $\lambda^2\eta^2-1=0$ and $\lambda^4-1=0$, $\lambda=\pm1$ and $\eta=\pm1$, those conditions give the matrix nullity two. In the case that both $\lambda^4-1=0$ and $\lambda^3\eta-1=0$, then $\lambda=\eta=\pm1$, which gives $K-I$ a nullity of three. A complete factorization shows that, there is a subspace of algebraic curvature tensors of dimension one preserved when any one of the following are satisfied: $\lambda-1=0$, $\lambda+1=0$, $\lambda\eta-1=0$, $\lambda\eta+1=0$, $\lambda^3\eta-1=0$, a subspace of dimension two when the pairs $\{\lambda-1=0,\ \lambda\eta+1=0\}$ and $\{\lambda+1,\ \lambda\eta-1=0\}$ are simultaneously satisfied, and a subspace of dimension three when the triples $\{\lambda-1=0,\ \lambda\eta-1=0,\ \lambda^3\eta-1=0\}$ and $\{\lambda+1=0,\ \lambda\eta-1=0$, and $\lambda^3\eta-1=0\}$ are satisfied. The following lattice can be constructed:



$^*\lambda=\eta=-1$
$^{**}\lambda=\eta=1$
$^{***}\lambda=1,\eta=-1$
$^{****}\lambda=-1,\eta=1$

In the bottom row of the lattice, when those factors are equal to zero, $K-I$ has a nullity of one. In these lattices, $N=$nullity of $K-I$.

In both of these cases, the possible nullities of $K-I$ are one, two, and three.

3.2 Classification

The classification of the Lie algebras of structure groups is displayed according to the Lie bracket relations and whether or not the given Lie algebra is semisimple. According to Theorem 3, the Lie algebra is semisimple if its killing form is non-degenerate, and the Lie bracket $[a,b]=ab-ba$ since the Lie algebra can be embedded into a ring of matrices. The categorization of the Lie algebras of structure groups is displayed according to the Lie bracket relations and whether or not the given Lie algebra is semisimple.

Obeidin [8] had classified the possible Lie algebras for structure groups of dimension three, the goal here is to better understand them by categorizing. He did this by reducing all but three of the curvature components to zero, the first case is when all three nonzero tensors are equal to one or only one of them is -1 and the other two are 1, the second case is where one of the components is zero, and the third is when all but one of the remaining three components is zero. In this paper, these known cases are categorized based on whether they are semisimple.

4 Results

4.1 Jordan Normal Forms

The following two subsections detail the findings of the cases where the matrix A has Jordan Normal forms $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$, and $J(2,a,b)\oplus J(1,\lambda)$. Referring to the methods section, this includes steps one, two, three, and four in that order.

4.1.1 Three Distinct Real Eigenvalues

For $A = J(1,\lambda) \oplus J(1,\eta) \oplus J(1,\tau)$, $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \tau \end{pmatrix}$ This means $Ae_1 = \lambda e_1; Ae_2 = \eta e_2; Ae_3 = \tau e_3$. So the relations are:

Since $A^*R_{1221} = R(\lambda e_1, \eta e_2, \eta e_2, \lambda e_1) = \lambda^2\eta^2 R(e_1, e_2, e_2, e_1)$,

$$A^*R_1 = \lambda^2\eta^2 R_1.$$

Since $A^*R_{1331} = R(\lambda e_1, \tau e_3, \tau e_3, \lambda e_1) = \lambda^2\tau^2 R(e_1, e_3, e_3, e_1)$,

$$A^*R_2 = \lambda^2\tau^2 R_2.$$

Since $A^*R_{2332} = R(\eta e_2, \tau e_3, \tau e_3, \eta e_2) = \eta^2\tau^2 R(e_2, e_3, e_3, e_2)$,

$$A^*R_3 = \eta^2\tau^2 R_3.$$

Since $A^*R_{1231} = R(\lambda e_1, \eta e_2, \tau e_3, \lambda e_1) = \lambda^2\eta\tau R(e_1, e_2, e_3, e_1)$,

$$A^*R_4 = \lambda^2\eta\tau R_4.$$

Since $A^*R_{2132} = R(\eta e_2, \lambda e_1, \tau e_3, \eta e_2) = \lambda\tau\eta^2 R(e_2, e_1, e_3, e_2)$,

$$A^*R_5 = \eta^2\lambda\tau R_5.$$

Since $A^*R_{3123} = R(\tau e_3, \lambda e_1, \eta e_2, \tau e_3) = \lambda^2\eta^2 R(e_3, e_1, e_2, e_3)$,

$$A^*R_6 = \tau^2\lambda\eta R_6.$$

Using these relations, one obtains this matrix:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} \lambda^2\eta^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda^2\tau^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2\tau^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^2\eta\tau & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2\lambda\tau & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau^2\lambda\eta \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix}, \text{and so}$$

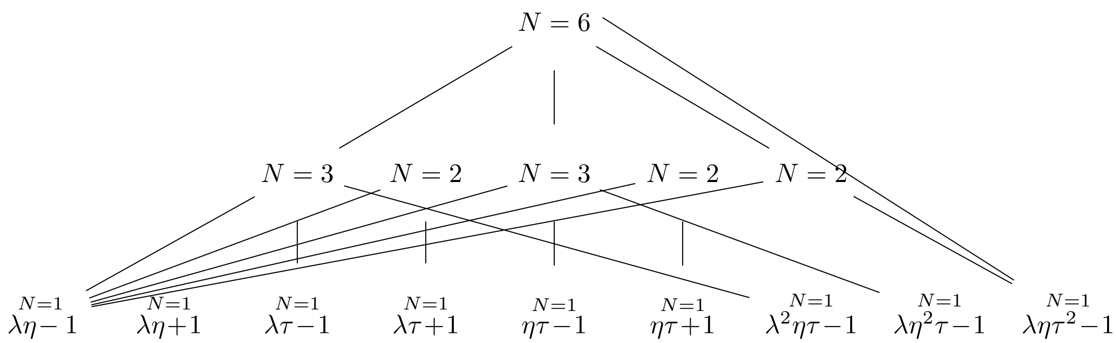
$$Det(K - I) = (\lambda^2\eta^2 - 1)(\lambda^2\tau^2 - 1)(\eta^2\tau^2 - 1)(\lambda^2\eta\tau - 1)(\eta^2\lambda\tau - 1)(\tau^2\lambda\eta - 1).$$

This can be factored into irreducible components:

$$Det(K - I) = (\lambda\eta - 1)(\lambda\eta + 1)(\lambda\tau - 1)(\lambda\tau + 1)(\eta\tau - 1)(\eta\tau + 1)(\lambda^2\eta\tau - 1)(\eta^2\lambda\tau - 1)(\tau^2\lambda\eta - 1).$$

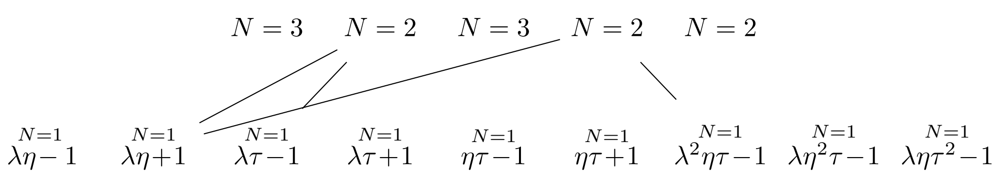
To find the dimension of the solution space of $K - I = 0$, find a condition for when $det(K - I) = 0$ is satisfied and compute the nullity of the Kaylor matrix given that condition. The dimension of the nullity of $K - I$ gives the dimension of the space of algebraic curvature tensors preserved by the matrix A. A lattice is constructed by putting irreducible factors of the polynomial $det(K - I)$ on the bottom row of the lattice. The higher levels of the lattice consist of situations when one or more of the bottom elements are satisfied at once, nullity is computed in the same way.

Below are two *incomplete* lattices, the first shows the irreducible factor $\lambda\eta - 1$ and combinations of that factor with other factors and the nullities of the matrix given several conditions including $\lambda\eta - 1$ are satisfied at once and the second similarly displays the factor $\lambda\eta + 1$. The remaining cases have not been filled in because the lattice would be difficult to read. By the symmetric nature of this case, the rest of the lattice follows in a similar way.



Since the Kaylor matrix is diagonal in this case, it is not difficult to see that each of the irreducible elements, when equal to zero, give the matrix $K - I$ at least nullity one. More specifically, when each of the bottom elements are individually satisfied, the nullity is one, when combinations of them are satisfied, the nullities N are given.

In the diagram above, it shows that when $\lambda\eta - 1 = 0$, $\lambda\tau - 1 = 0$, and $\lambda^2\eta\tau - 1 = 0$ are satisfied at once ($\lambda\eta - 1 = 0$ and $\lambda\tau - 1 = 0$ implies $\lambda^2\eta\tau - 1 = 0$), $K - I$ has nullity three. When $\lambda\eta - 1 = 0$ and $\lambda\tau + 1 = 0$ are both satisfied, $K - I$ has nullity two, and so on.



This diagram shows that when $\lambda\eta + 1 = 0$ and $\lambda\tau - 1 = 0$ are both satisfied, the matrix has nullity two and when $\lambda\eta + 1 = 0$ and $\lambda^2\eta\tau - 1 = 0$ are satisfied at once, the matrix has nullity two. There are no higher levels of this lattice because no two elements from the second row can be satisfied at once.

4.1.2 One Real and Two Complex Conjugate Eigenvalues

For $A = J(2,a,b) \oplus J(1,\lambda)$, $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix}$. This means $Ae_1 = \lambda e_1$; $Ae_2 = ae_2 - be_3$; $Ae_3 = be_2 + ae_3$. So the relations are:

Since $A^*R_{1221} = R(\lambda e_1, ae_2 - be_3, ae_2 - be_3, \lambda e_1)$
$= R(\lambda e_1, ae_2, ae_2, \lambda e_1) + R(\lambda e_1, ae_2, -be_3, \lambda e_1) + R(\lambda e_1, -be_3, ae_2, \lambda e_1) + R(\lambda e_1, -be_3, -be_3, \lambda e_1)$
$= a^2\lambda^2 R_{1221} - 2\lambda^2 ab R_{1231} + \lambda^2 b^2 R_{1331}$,

$$A^*R_1 = a^2\lambda^2 R_1 + b^2\lambda^2 R_2 - 2ab\lambda^2 R_4.$$

Since $A^*R_{1331} = R(\lambda e_1, be_2 + ae_3, be_2 + ae_3, \lambda e_1)$
$= R(\lambda e_1, be_2, be_2, \lambda e_1) + R(\lambda e_1, be_2, ae_3, \lambda e_1) + R(\lambda e_1, ae_3, be_2, \lambda e_1) + R(\lambda e_1, ae_3, ae_3, \lambda e_1)$
$= b^2\lambda^2 R_{1221} + 2\lambda^2 ab R_{1231} + \lambda^2 a^2 R_{1331}$,

$$A^*R_2 = b^2\lambda^2 R_1 + a^2\lambda^2 R_2 + 2ab\lambda^2 R_4.$$

Since $A^*R_{2332} = R(ae_2 - be_3, be_2 + ae_2, be_2 + ae_2, ae_2 - be_3)$
$= R(ae_2, ae_3, be_2, -be_3) + R(ae_2, ae_3, ae_3, ae_2) + R(-be_3, be_2, ae_3, ae_2) + R(-be_3, be_2, be_2, -be_3)$
$= a^2b^2 R_{2332} + a^4 R_{2332} + a^2b^2 R_{2332} + b^4 R_{2332}$
$= (a^2 + b^2)^2 R_{2332}$,

$$A^*R_3 = (a^2 + b^2)^2 R_3.$$

Since $A^*R_{1231} = R(\lambda e_1, ae_2 - be_3, be_2 + ae_3, \lambda e_1)$
$= R(\lambda e_1, ae_2, be_2, \lambda e_1) + R(\lambda e_1, ae_2, ae_3, \lambda e_1) + R(\lambda e_1, -be_3, be_2, \lambda e_1) + R(\lambda e_1, -be_3, ae_3, \lambda e_1)$
$= ab\lambda^2 R_{1221} + a^2\lambda^2 R_{1231} - b^2\lambda^2 R_{1231} - ab\lambda^2 R_{1331}$,

$$A^*R_4 = ab\lambda^2 R_1 - ab\lambda^2 R_2 + \lambda^2(a^2 - b^2)R_4.$$

Since $A^*R_{2132} = R(ae_2 - be_3, \lambda e_1, be_2 + ae_3, ae_2 - be_3)$
$= R(ae_2, \lambda e_1, be_2, -be_3) + R(ae_2, \lambda e_1, ae_3, ae_2) + R(-be_3, \lambda e_1, be_2, -be_3) + R(-be_3, \lambda e_1, ae_3, ae_2)$
$= b^2a\lambda R_{2132} + a^3\lambda R_{2132} + b^3\lambda R_{3123} + a^2b\lambda R_{3123}$
$= \lambda a(b^2 + a^2)R_{2132} + \lambda b(b^2 + a^2)R_{3123}$,

$$A^*R_5 = a\lambda(b^2 + a^2)R_5 + b\lambda(a^2 + b^2)R_6.$$

Since $A^*R_{3123} = R(be_2 + ae_3, \lambda e_1, ae_2 - be_3, be_2 + ae_3)$
$= R(be_2, \lambda e_1, ae_2, ae_3) + R(be_2, \lambda e_1, -be_3, be_2) + R(ae_3, \lambda e_1, ae_2, ae_3) + R(ae_3, \lambda e_1, -be_3, be_2)$
$= -a^2b\lambda R_{2132} - b^3\lambda R_{2132} + a^3\lambda R_{3123} + b^2a\lambda R_{3123}$

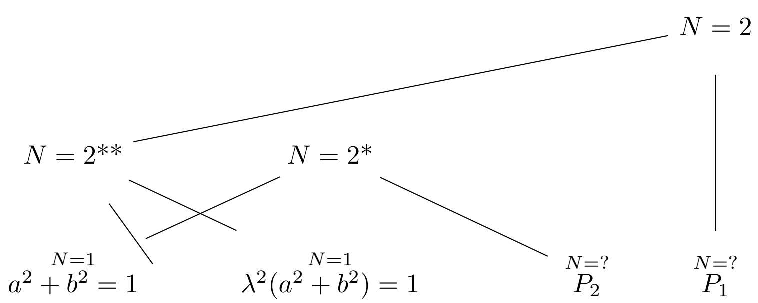
$= -b\lambda(a^2+b^2)R_{2132} + a\lambda(a^2+b^2)R_{3123},$

$$A^*R_6 = -b\lambda(a^2+b^2)R_5 + a\lambda(a^2+b^2)R_6.$$

One obtains the following matrix:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix} = \begin{pmatrix} a^2\lambda^2 & b^2\lambda^2 & 0 & ab\lambda^2 & 0 & 0 \\ b^2\lambda^2 & a^2\lambda^2 & 0 & -ab\lambda^2 & 0 & 0 \\ 0 & 0 & (a^2+b^2)^2 & 0 & 0 & 0 \\ -2ab\lambda^2 & 2ab\lambda^2 & 0 & (a^2-b^2)\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a\lambda(a^2+b^2) & -(a^2+b^2)b\lambda \\ 0 & 0 & 0 & 0 & b\lambda(a^2+b^2) & a\lambda(a^2+b^2) \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \end{pmatrix}$$

$$Det(K-I) = (-1+a^2+b^2)(1+a^2+b^2)(P_1)(-1+a^2\lambda^2+b^2\lambda^2)(P_2)$$
$$P_1 = (1+4ab\lambda^4-2a^2\lambda^2+2b^2\lambda^2+a^4\lambda^4-2a^2b^2\lambda^4+b^4\lambda^4)$$
$$P_2 = (1-2a^2a\lambda+a^4a\lambda^2-2a\lambda b^2+2a^2a\lambda^2b^2+a\lambda^2b^4+a^4b\lambda^2+2a^2b^2b\lambda^2+b^4b\lambda^2)$$



$*\ \lambda^2=1, a=0, b=-1$, $**\lambda^2=1, a^2+b^2=1$

In the bottom row of the lattice, when $a^2+b^2=1$ and $\lambda^2(a^2+b^2)=1$ are satisfied, $K-I$ has a nullity of one. The nullities when P_1 and P_2 are equal to zero is not known. There are no higher levels of this lattice because no two elements in the second row can be satisfied at once.

This completes the analysis for all four Jordan Normal forms.

Proposition 4. *There does not exist a matrix in Jordan Normal Form such that the Nullity of $K-I=4$ or 5 for algebraic curvature tensors over a vector space of dimension three.*

4.2 Expressing one-dimensional null spaces in terms of R_φ

In the cases of nullity one, it is possible to find an algebraic curvature tensor which spans the space preserved. There is a theorem by Diaz-Ramos and Garcia-Rio [5] that explains different constructions of curvature tensors, based on properties of the associated Ricci tensor. The two cases examined in this paper involve expressing the tensor that spans the solution space in terms of one R_φ and as a linear combination of an R_{φ_1} and an R_{φ_2}.

The process involves looking at which curvature tensor entries can be nonzero to be in the kernel of $K-I$, and using the property that $g(\Phi e_i, e_j) = \rho(e_i, e_j)$, where g is a metric defined so that $g_{ij} = g(e_i, e_j) = 0$, $i \neq j$ and $g_{ii} = g(e_i, e_i) = 1$ and $\rho(x, y)$ is the Ricci tensor, defined by $\rho(x,y) = \sum_{i=1}^{n} R(x, e_i, e_i, y)$. Below, the known cases are listed:

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\eta=1$, the only non-zero entry of the tensor is R_1, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\eta=-1$, the only non-zero entry of the tensor is R_1, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\tau=1$, the only non-zero entry of the tensor is R_2, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\tau=-1$, the only non-zero entry of the tensor is R_2, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\eta\tau=1$, the only non-zero entry of the tensor is R_3, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\eta\tau=-1$, the only non-zero entry of the tensor is R_3, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda^2\eta\tau=1$, the only non-zero entry of the tensor is R_4, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\eta^2\tau=1$, the only non-zero entry of the tensor is R_5, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(1,\eta)\oplus J(1,\tau)$ when $\lambda\eta\tau^2=1$, the only non-zero entry of the tensor is R_6, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(2,a,b)$ when $-1+a^2\lambda^2+b^2\lambda^2=0$, the first and second rows of $K-I$ with this condition are multiples of one another, so R_1 and R_2 are the only nonzero entries for the tensor.
  Then, R, the tensor that spans the one dimensional space, is $R=k(-R_{\varphi_1}-R_{\varphi_2})$ where k is a scalar

  and $R_{\varphi_1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $R_{\varphi_2}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

- $J(1,\lambda)\oplus J(2,a,b)$ when $-1+a^2+b^2=0$, the only non-zero entry of the tensor is R_3, and in this case $R=kR_\varphi$, where k is a scalar and R_φ has $\varphi=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

4.3 Classification

The three nontrivial cases for the structure groups of dimension three must have Lie group dimension three, four, and six. The following are elements in the Lie algebra of the structure groups: [8]

$$\text{Dimension 3} \begin{pmatrix} 0 & x_1 & -x_2 \\ -x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} \text{ (i.e. } \mathfrak{so}(3)) \text{ ; } \begin{pmatrix} 0 & -x_3 & x_2 \\ -x_3 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \text{ (i.e. } \mathfrak{so}(1,2))$$

$$\text{Dimension 4} \begin{pmatrix} x_1 & 0 & 0 \\ x_3 & -x_1 & x_2 \\ x_4 & -x_2 & -x_1 \end{pmatrix}$$

$$\text{Dimension 6} \begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & -x_1 & 0 \\ x_4 & x_5 & x_6 \end{pmatrix}$$

Dimension 3: The Lie algebra $\mathfrak{so}(1,2)$ or $\mathfrak{so}(3)$.
Dimension 4: Lie bracket relations $[e_1,e_3]=-2e_3; [e_2,e_3]=-e_4; [e_1,e_4]=-2e_4; [e_2,e_4]=e_3$.
Dimension 6: Lie bracket relations $[e_5,e_6]=-e_5; [e_3,e_5]=-e_4; [e_4,e_6]=-e_4; [e_2,e_4]=-e_5;$ $[e_2,e_3]=e_1; [e_1,e_5]=e_5; [e_1,e_4]=-e_4; [e_1,e_3]=-2e_3; [e_1,e_2]=2e_2$.

Proposition 5. *If $dimV=3$, then there are only two semisimple Lie algebras corresponding to any G_R for $R\in\mathcal{A}(V)$. These are isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{so}(1,2)$.*

5 Conclusion

The remaining two cases of the Jordan Normal forms have been examined, in sum, the only nullities that did not appear were four or five (Proposition 4). The known Lie algebras of the structure groups of algebraic curvature tensors have been classified, and given a Lie group, if its Lie algebra is semisimple but not isomorphic to $\mathfrak{so}(3)$ or $\mathfrak{so}(1,2)$, then it is known that the group is not the structure group of an algebraic curvature tensor in dimension three.

6 Open Questions

1. Find relationships between Jordan Normal forms used in the first part of the paper and the known structure groups, are any of these Jordan Normal forms in the known structure groups?
2. Find properties of Lie algebras of Lie groups which are structure groups of an algebraic curvature tensor in dimension n (simple, semisimple, abelian).
3. Construct lattices for dimension four (and higher).
4. Examine lattices and their different levels, why sometimes combining m components from the bottom row gives a nullity of $K-I$ that is equal to n where $n\neq m$.
5. Know, out of curvature tensors in three dimensional space which subgroups of GL_3 are structure groups of algebraic curvature tensors.
6. When is the lattice disconnected?
7. When is the Kaylor matrix symmetric/skew-symmetric?

7 Acknowledgements

I would like to thank Dr. Dunn, Dr. Trapp, and Dr. Insko for their help, support, and advice throughout the program. I am particularly grateful to Dr. Dunn for his excellent and substantial guidance; working with him has been a tremendous learning experience. This research was jointly funded by NSF grant DMS-1156608, and by California State University, San Bernardino. Any remaining errors are my own.

References

[1] Bowers, Adam "Classification of Three-Dimensional Real Lie Algebras." 2005.

[2] Diaz-Ramos, Carlos; Garcia-Rio, Eduardo "A note on the structure of algebraic curvature tensors." Elsevier Linear Algebra and its Applications 2003.

[3] Gilkey, Peter, *Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor*, World Scientific, New Jersey, 9 (2001).

[4] Humphreys, James, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 22 (1972).

[5] Helgason, Sigurdur, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 99 (1978).

[6] Kaylor, Lisa "Is Every Invertible Linear Map in the Structure Group of some Algebraic Curvature Tensor?" CSUSB REU Program 2012.

[7] Klinger, Ruedi "A Basis that Reduces to Zero as many Curvature Components as Possible." Abh. Math. Sem. Univ. Hamburg 61 (1991), 243-248.

[8] Obeidin, Malik "On the Computation and Dimension of Structure Groups of Algebraic Curvature Tensors" CSUSB REU Program 2012.