Torified Rational Links

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Abstract

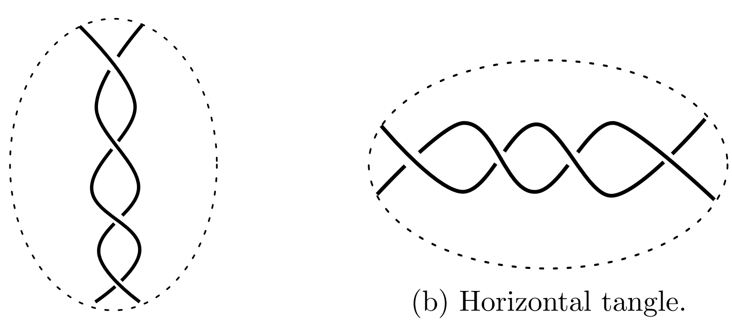
This paper examines a certain class of links, called Torified Rational Links, with a focus on bounds for stick number. These bounds are found by supercoiling the Torified Rational tangles and attaching the coil to an outer skeleton, similar to that of a rational link. The bounds obtained using this model are compared to the known upper bound for the stick number of any link, $s(L) \leq \frac{3}{2}(c(L)+1)$, using relationships between crossing number and the maximal and minimal degrees of the variables in the HOMFLY polynomial.

Introduction

In this section, we define the class of rational links, which we use as a model for constructing Torified Rational Links. Then we supply the necessary background information on stick number, supercoils, and the HOMFLY polynomial.

Constructing Torified Rational Links

A tangle is considered to be rational if it can be untangled by moving its endpoints around the boundary of a sphere. All rational tangles are made up of a collection of integral tangles. Integral tangles are simply two strands twisted together (see Figure 1).



(a) Vertical tangle.

(b) Horizontal tangle.

Figure 1: Examples of two kinds of integral tangles.

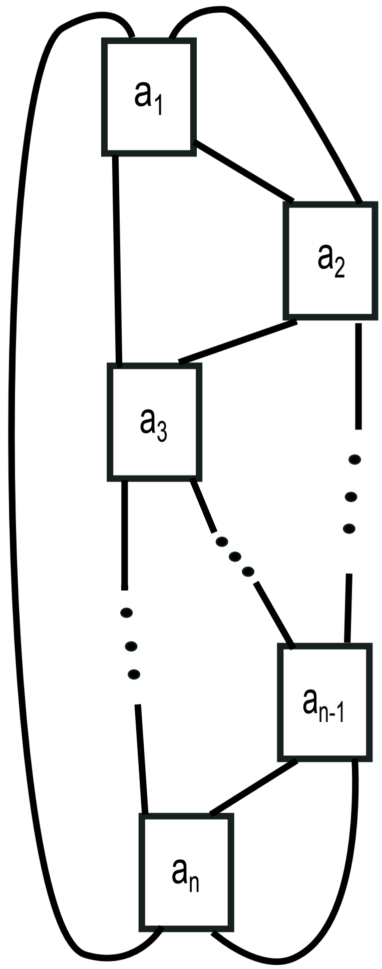


Figure 2: Rational Link with n twist-boxes.

Each integral tangle can have any number of crossings, as long as all of the twists are of the same type (i.e. all horizontal or all vertical). Stringing vertical integral tangles together in a particular way, as is shown in Figure 2, produces a rational link. A rational link consists of twist-boxes, which each contain an integral tangle, and an outer skeleton.

Definition 1. *The* ***skeleton*** *of a link is the collection of strands that connect its twist-boxes.*

In this paper, we are going to examine a similar class of links which can be created by swapping out the integral tangles of rational links for a different kind of tangle, while preserving, up to parallel cables , the skeletal structure of rational links. We will call such links Torified Rational Links. To describe the tangles in the twist-boxes of Torified Rational Links, we will use σ moves.

Definition 2. *A* σ ***move****, which we denote* $\sigma(12\cdots n-1)$*, moves the rightmost input strand across all other strands (Figure 3).*

In each twist box of a Torified Rational Link, we have a $\sigma^x(12\cdots n-1)$ tangle, where x is the number of times that a strand is pulled across all of the other strands. Each σ move contributes $n-1$ crossings to the diagram.

Figure 4 is an example of a Torified Rational Link of two components. It has three twist-boxes which contain $\sigma^3(1234)$, $\sigma^{-1}(1234)$, and $\sigma^4(1234)$ tangles, respectively. Consider the first tangle (top). The number of strands coming out of the bottom are split: two go to the left and three go to the right. If we changed the number of strands going each way, then we might get a different link.

For example, consider the two Torified Rational Links shown in Figure 5. Their twist-boxes are identical, but their skeletons are different. The first link's strands are broken up with three going left and two going right. It is, in fact, a knot (a link of one component). However, the second link, which has four strands going to the left and only one going to the right, has two components. Thus, the two seemingly similar Torified Rational Links are clearly not the same.

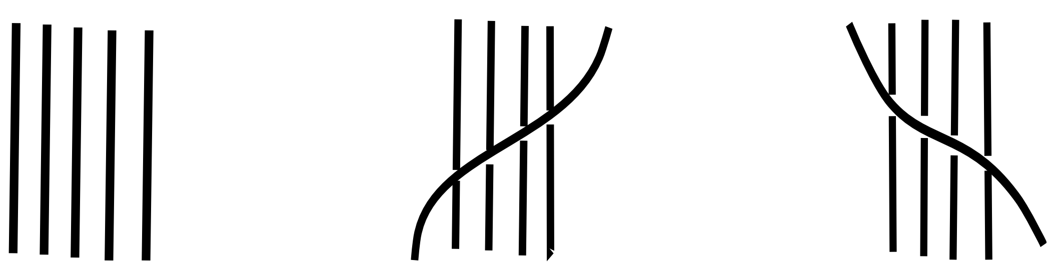
An Introduction to Supercoiling

Many invariants can help to distinguish and define links. These include link and knot polynomials, covers, colorings, etc. One such measure is called the stick number.

Definition 3. *The* ***stick number****, $s(L)$, of a link L is the minimum number of line segments (sticks) needed to represent L. The sticks must be non-intersecting, and can only join with one another at a common vertex.*

In short, stick number is a measure of the complexity of a given link. Huh and Oh have shown that, for any link L, $s(L) \leq \frac{3}{2}(c(L)+1)$, where $c(L)$ is the minimal crossing number of the link [6]. Improved bounds on stick number have been found for many individual classes of links. Most recently, Insko and Trapp found a way to significantly improve this bound for sufficiently complex 2-bridge links. They used the shape and structure of DNA supercoils as a model for a non-minimal crossing polygonal projection of rational tangles (Figure 6). The tangle in Figure 6 represents an alternating, 13-crossing tangle with 12 sticks. Then, for every four sticks added, we get six more crossings [7].

We can use a similar model to construct a polygonal representation of twist boxes of Torified Rational Links. All we need to do differently is add in more strands, as shown in Figure 7. Notice



(a) Identity strands. (b) $\sigma(1234)$. (c) $\sigma^{-1}(1234)$.

Figure 3: Example of σ moves using five strands.

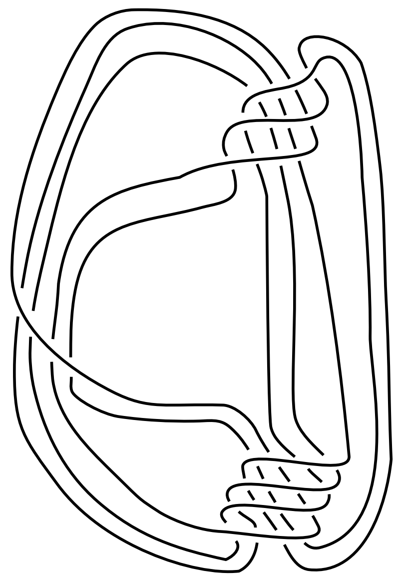
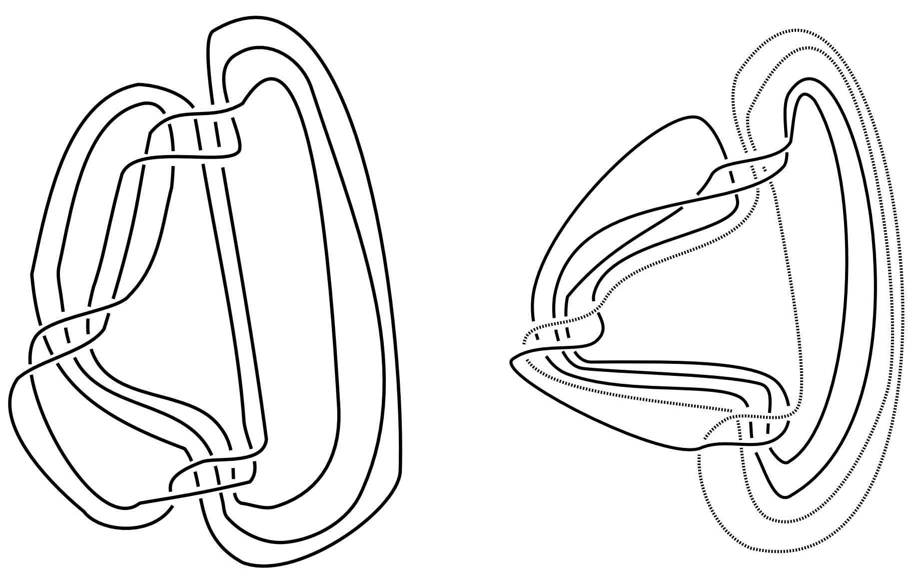


Figure 4: Torified Rational Link.



(a) (b)

Figure 5: Two distinct Torified Rational Links.

that the supercoiling model gets more efficient as the number σ moves increases. The supercoiled tangle shown in Figure 7 results in ten σ moves: five from the corners (Figures 7 (a) & (b)), and five from the other crossings. The fact that such tangles can be realized geometrically has been proven [8].

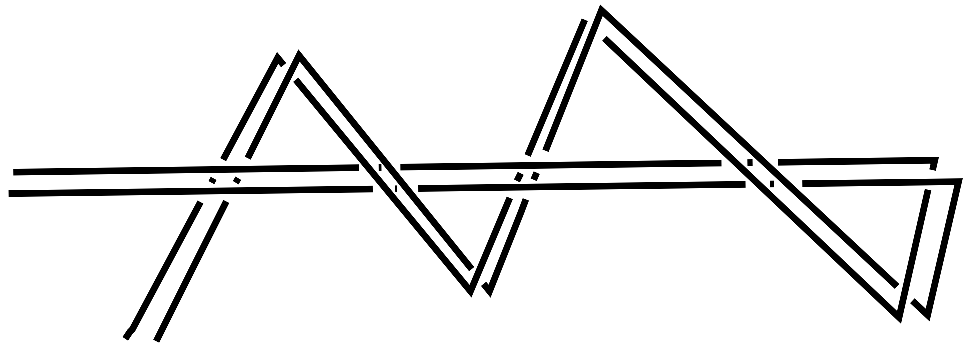
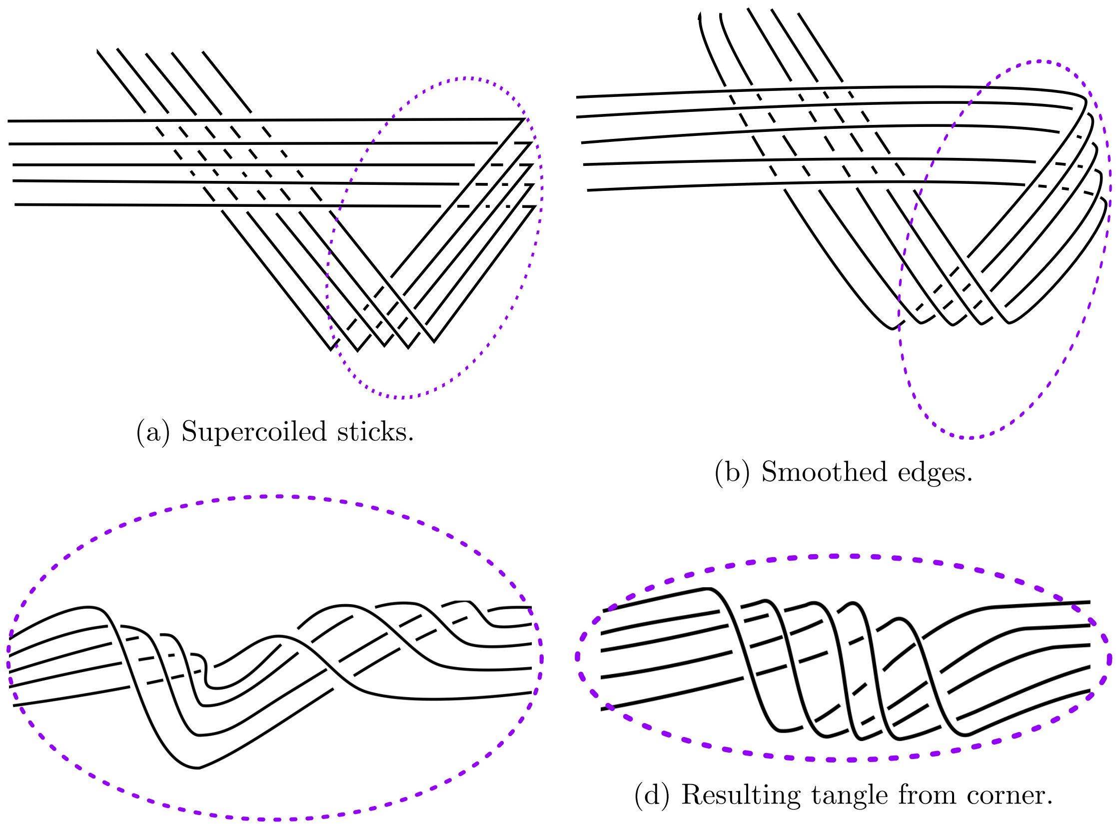


Figure 6: Supercoil of a 2-bridge link's rational tangle.



(a) Supercoiled sticks.

(b) Smoothed edges.

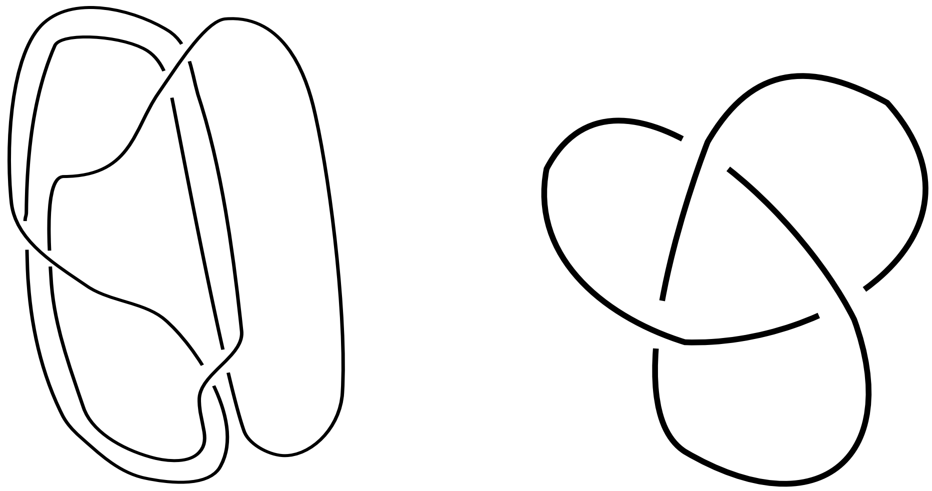
(c) Corner pulled straight.

(d) Resulting tangle from corner.

Figure 7: Simple supercoil using five strands.

The HOMFLY Polynomial and Minimal Crossing Number

In order to compare our bounds on stick number to that of Huh and Oh, we need to consider the minimal crossing number, $c(L)$, of our Torified Rational Link, L. We know that alternating diagrams are of minimal crossing number. The difficulty for us lies in the fact that we are not dealing with alternating diagrams. There are many Torified Rational Link diagrams that make the link seem far more complicated than it actually is. For example, Figure 8 shows a Torified Rational Link with three twist boxes, which has 6 crossings, that is isotopic to the left-handed trefoil, which has as few as 3 crossings. It isn't obvious, at first glance, that this Torified Rational Link is merely a trefoil.



(a) Representation as a Torified Rational Knot with three twist-boxes.

(b) Minimal crossing representation.

Figure 8: Two diagrams of the left-handed trefoil.

To address this problem, we need to use Gruber's bound for $c(L)$, which can be derived from the HOMFLY polynomial of L. That is, for every link L,

$$c(L) \geq M + \frac{1}{2}(E - e), \tag{1}$$

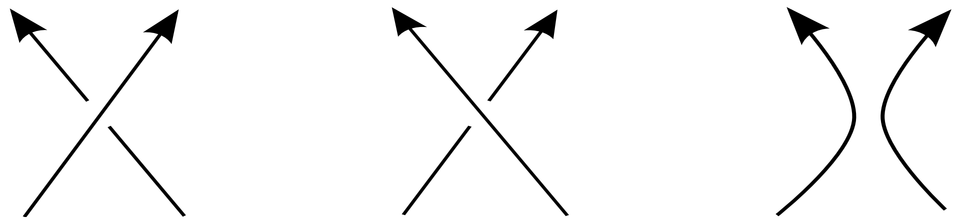
where M is the maximal non-zero exponent on z, E is the maximal non-zero exponent on v, and e is the minimal non-zero exponent on v [4]. Naturally, the crossing number will not depend on the orientation of our Torified Rational Links, so we can arbitrarily orient them to find bounds on the crossing numbers of these unoriented links. We will use this method to determine wether or not our bound on stick number found using supercoils is superior to that of Huh and Oh.

The HOMFLY polynomial is a two-variable polynomial that is defined using skein relations on oriented knots and links.

Definition 4. *Consider three oriented links, L_+, L_-, and L_0. Let them be exactly the same except in the neighborhood of one crossing, where they appear as in Figure 9. Then, L_+, L_-, and L_0 are called a **skein triple**.*

The HOMFLY polynomial is given by:

$$P(\text{unknot}) = 1,$$



(a) L_+ (b) L_- (c) L_0

Figure 9: Skein Triple.

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z). \tag{2}$$

Additionally, it is important to realize that a bound on crossing number that does not rely on knowing the minimal crossing number of a link is useful, even if it is not always better than the other known bounds, because often we are presented diagrams of complex links, and the value of $c(L)$ is not evident and very difficult to calculate.

Results

In this section we will look at bounds on stick number for Torified Rational Links with three twist-boxes. Then we will compare our findings to the bound given by Huh and Oh. Finally, we generalize our results for Torified Rational Links with n twist-boxes.

Links with three twist-boxes

We will start by looking at Torified Rational Links that have three twist-boxes, like the one shown in Figure 4. These simple cases of Torified Rational Links are constructed as in Figure 10, where the w_i represent the weights of the strands, i.e. the number of strands along that edge of the link's skeleton. Numerous relationships and symmetries occur when we set the number of twist-boxes to three.

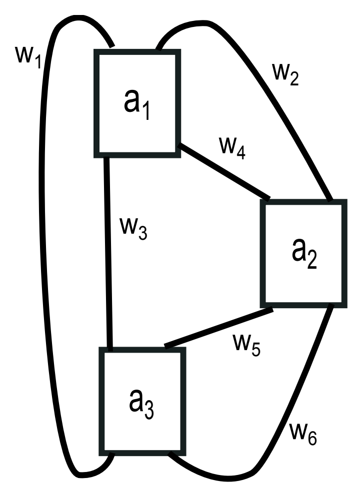


Figure 10: The structure of Torified Rational Links with three twist-boxes.

Claim 1. *The weights of each section of the skeleton of a three-box Torified Rational Link, $w_1, w_2, ..., w_6$, are completely determined by w_1 and w_2.*

Proof. First, observe that the number of strands going into a twist-box must equal the number of strands coming out of it. Therefore, boxes a_1, a_2, and a_3, respectively, give us the system of equations:

$$w_1 + w_2 = w_3 + w_4,$$
$$w_2 + w_4 = w_5 + w_6,$$
$$w_3 + w_5 = w_1 + w_6.$$

It follows that $w_1 = w_4 = w_5$, and $w_2 = w_3 = w_6$. Therefore the two weights, w_1 and w_2, completely determine the weight of each component of the link's skeleton. □

From the relationship described in Claim 1, we immediately discover a number of interesting symmetries of Torified Rational Links with three twist-boxes. One such result is described in Corollary 1.

Corollary 1. *For any diagram, D, of a Torified Rational Link, L, with three twist-boxes, and weights $w_1 = a$ and $w_2 = b$, we can find an isotopic diagram, D', with the same number of crossings in each twist-box and with $w_1 = b$ and $w_2 = a$.*

Proof. Let's start with an arbitrary diagram, D, of a Torified Rational Link, L, with three twist-boxes and weights $w_1 = a$ and $w_2 = b$, as shown in Figure 11.

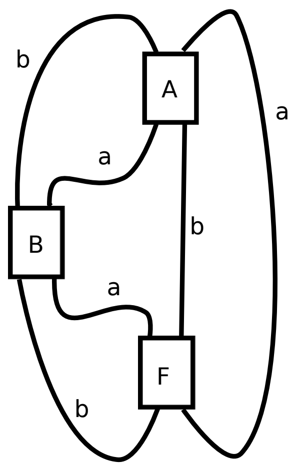
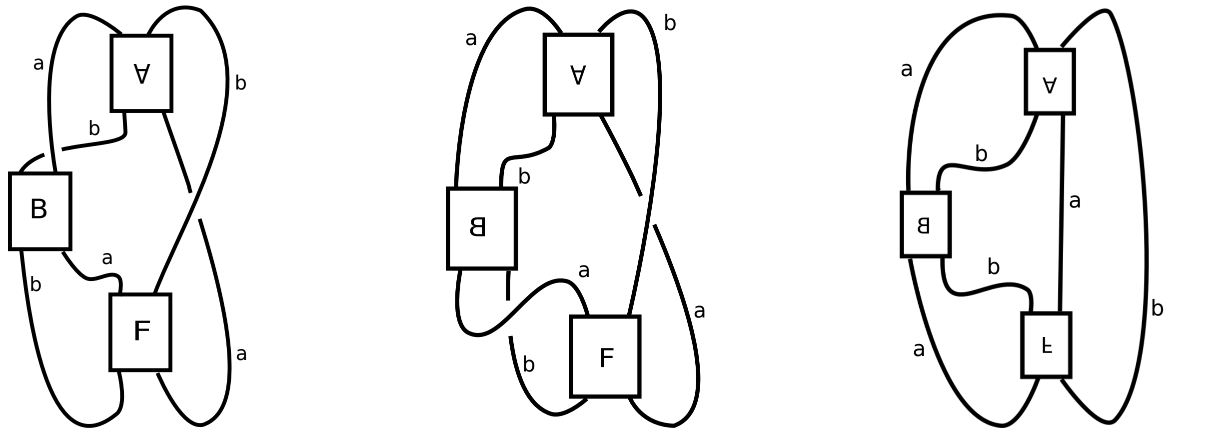


Figure 11: Diagram D.

Then we can do the following series of isotopies of L:



(a) Flip box A vertically. (b) Flip box B horizontally. (c) Flip box F vertically to get D’

Figure 12: Isotopies that transform D into D'.

First notice that the twist-boxes have merely been flipped, so they still contain the same number of crossings. In fact, they contain the same number of σ moves, but with σ^{-1} moves instead of σ moves in all three boxes. Thus by following the isotopies, we have found that for any diagram, D, we have an equivalent diagram, D', with the same number of crossings in each twist box and with w_1 and w_2 weights reversed. $\square$

We will use the results from Claim 1 and Corollary 1 to find a bound for the stick number of the skeleton of our Torified Rational Link.

Lemma 1. *Let $a = min\{w_1, w_2\}$, $b = max\{w_1, w_2\}$, and let $a + b = n$. Then the number of sticks needed to build the skeleton of a Torified Rational Link, L, with three twist-boxes, $s_s(L)$, is bounded by:*

$$s_s(L) \leq 4n + a. \tag{3}$$

Proof. Consider the diagram of an arbitrary Torified Rational Link with three twist-boxes, where $a = b = 1$, as shown in Figure 13. We will show that this model of the skeleton can be constructed in three-space without self-intersection, and in such a way that it will not intersect with the tangles, which are completely contained inside the twist-boxes. When we construct the twist-boxes, the input and output strands will connect through opposite sides of a twist-box.

In the model shown in Figure 13, let the bases of the three twist-boxes be coplanar. It is apparent that the three central sticks can exist in three-space without intersecting each other, and only intersect the twist-boxes at a point. For our purposes, intersection with a twist-box at a point is a trivial intersection. The two outer a strands can also exist in three-space without intersecting each other and without nontrivially intersecting the twist-boxes. The only strand left to consider is the outer b strand, which does intersect two twist-boxes non-trivially. These regions of intersection are circled in Figure 13.

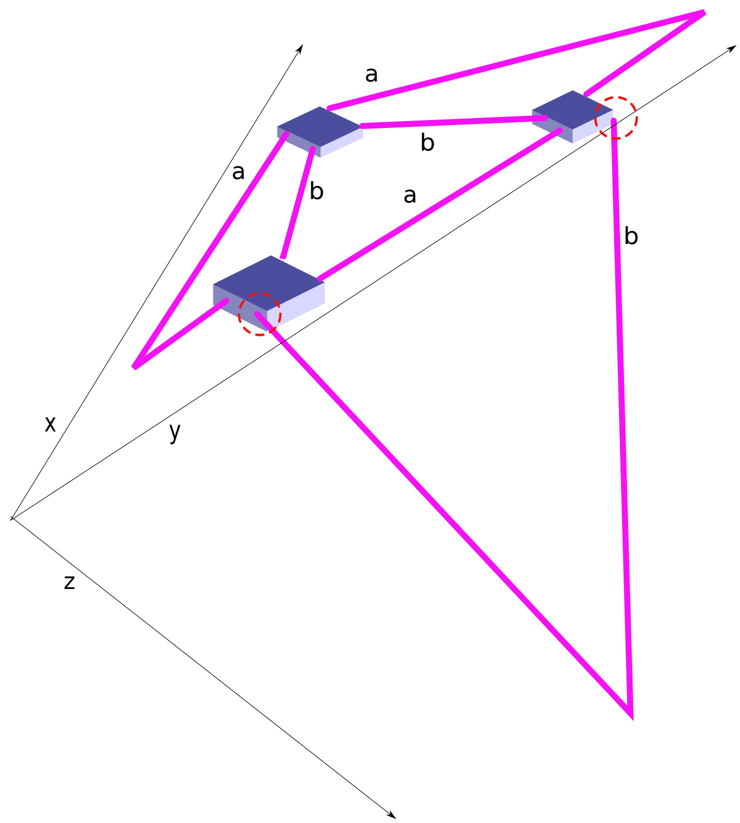


Figure 13: Skeleton of a Torified Rational Link with three twist-boxes made from sticks.

We can choose a point at the lower right far enough away so that the encircled regions of intersection occur within an angle of ϵ of the face of the twist-box. Therefore, though that portion of the skeleton does have non-trivial intersection with two boxes, it can still exist in three-space without intersecting the tangles in the boxes. Thus the skeleton of this link can be made from 9 sticks.

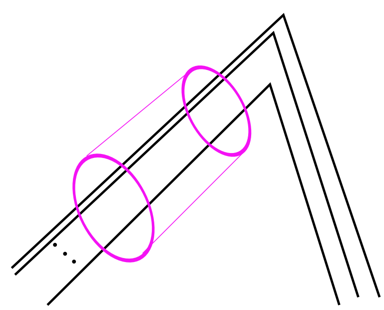


Figure 14: Increasing strand number.

Now we can generalize this for any weights a and b. Imagine that the skeleton shown in Figure

13 is not a collection of sticks, but of tubes. Then each tube can contain as many strands as we want without intersection (see Figure 14).

Recall Corollary 1, which tell us that we can let $a \leq b$, since if it is not, then we can represent the same link with a diagram where a and b are swapped. Note that other parts of the diagram may be flipped as well, but this doesn't effect the skeleton. This means that, without loss of generality, we can construct the skeleton with $4b + 5a$ sticks by constructing it as in Figure 13. Finally, we know that $a + b = n$; so it follows that $s_s(L) \leq 4n + a$. □

Next we need to construct a bound for the number of sticks needed to create the Torified Rational tangles $s_T(L)$. These will be clipped into the skeleton constructed in Lemma 1.

Claim 2. *In $\mathbb{R}^3$, we can represent any twist box of n strands containing x σ moves, where $|x| \geq 1$, with $2(|x| - 1) + n$ sticks or fewer.*

Proof. Suppose $x > 0$. As shown in Figure 15, if $x = 1$, we can construct our tangle with n sticks.

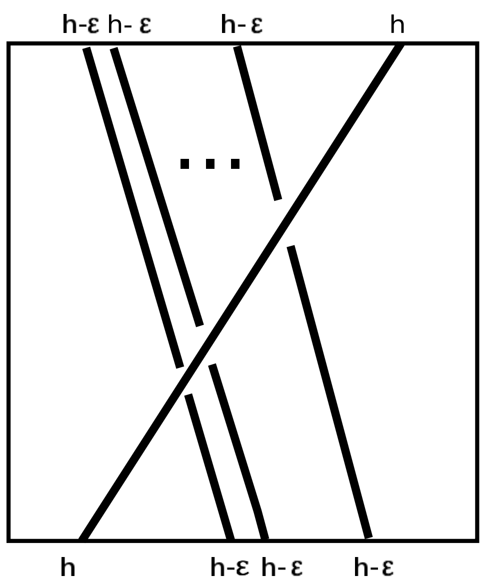


Figure 15: The stick representation of a Torified Rational tangle with one σ move.

Then, as we add on more σ moves, we can simply pile on similar boxes (see Figure 16). In this model, we have line segments that go from height $h - \epsilon$ to h crossing over some segments which go from height h to $h - \epsilon$. We can choose the angle to drop down the overcrossing strand such that is is above the other when it reaches the crossing. Thus we can build this model in three-space.

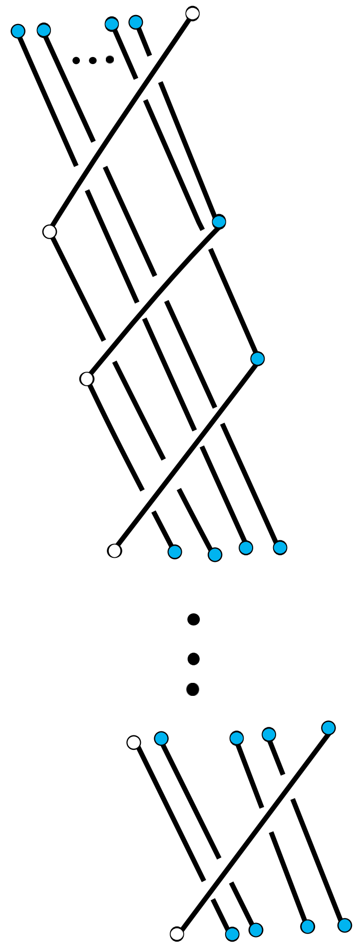


Figure 16: The stick representation of a Torified Rational tangle with x σ moves. The white vertices are at height h, and the shaded vertices are at height $h - \epsilon$.

Notice that each subsequent σ move requires the addition of two new sticks, which supports our assumption about $s_t(L)$. The formula $s_t(L) \leq 2(|x| - 1) + n$ directly follows from this model.

The same argument can be used for $x < 0$. $\square$

The above proof accounts for the cases where $|x| > 0$. For the trivial case of $x = 0$, we would simply need n sticks.

Keeping Claim 2 in mind, we will consider supercoiling our tangles in order to see what kind of bounds we can get on stick number that way. Ultimately, we will combine the two methods to achieve an optimal bound.

Claim 3. *We can represent any Torified Rational tangle, T, with $(2+3k)n$ σ moves using $(3+2k)n$ sticks or fewer, where n is the number of strands and $k \in \mathbb{Z}$.*

Proof. Consider Figure 17. We know that two of the B regions correspond to n σ moves. This is shown in more detail in Figure 7. Furthermore, each A region corresponds to n σ moves. That much is evident; just imagine pulling the ends tight. Then the red regions would each add a complete twist, which is n σ moves.

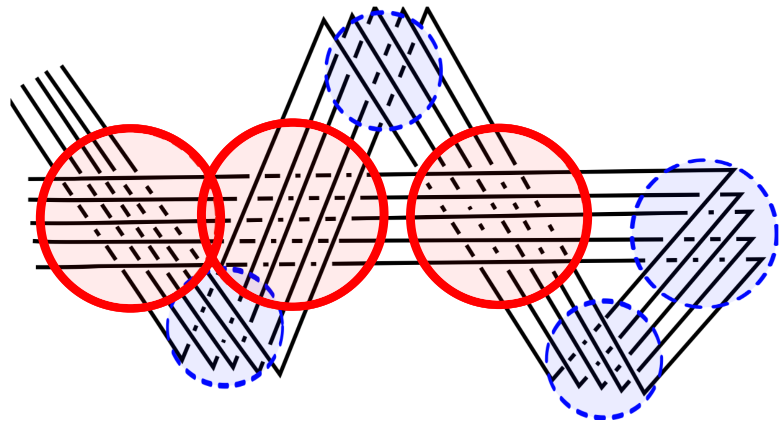


Figure 17: Crossing contribution of a supercoiled Torified Rational tangle (A regions circled with solid lines, B regions circled with dotted lines).

The rest of the proof follows from the Figure. It takes $3n$ sticks to create the simplest supercoil, and then for each additional $2n$ sticks, we get B regions and two A regions, which represents $3n$ σ moves. Hence $(2+3k)n$ σ moves can be modeled using $(3+2k)n$ sticks. $\square$

For the sake of clarity, in following argument we will assume that we are doing positive σ moves, as opposed to inverse σ moves. The argument, however, works for both cases.

Lemma 2. *Let T be a Torified Rational tangle of some Torified Rational Link, L. Let T have x σ moves and n strands. Choose k to be the largest integer such that $x = (2+3k)n + y$, and $y \geq 0$. Then,*

$$s_T(L) \leq \begin{cases} (3+2k)n + 2(y-1) & \text{if } y \leq 3n(k+1) \\ (5+2k)n + 2(x-y-1) & \text{if } y > 3n(k+1). \end{cases} \tag{4}$$

Proof. We have two options for how to create our stick representation of T: we can make a supercoil and then add on the extra tangles using the method described in Claim 2, or we can make a larger supercoil and then undo some of the tangles by that same method, but with σ^{-1} moves. It is easy enough to compile these two moves since both allow for controlling the positions of the incoming line segments. For the first option, we would need $(3+2k)n$ sticks to make the coil, and then $2(y-1)$ sticks to make the remaining σ moves. This gives us

$$(3+2k)n + 2(y-1) \tag{5}$$

sticks. For the other option, we would need $(3+2(k+1))n$ sticks to make the supercoil, and then $2(x-y-1)$ additional sticks to undo the superfluous σ moves. This gives us

$$(5+2k)n + 2(x-y-1) \tag{6}$$

total sticks.

Now all we need to do is show when it is optimal to use Equation 5 and when it is better to use Equation 6. We should use Equation 5 when

$$\begin{aligned}(3+2k)n+2(y-1) &\leq (5+2k)n+2(x-y-1)\\ 2y-2 &\leq 2n+2x-2y-2\\ y &\leq 3n(k+1)\end{aligned}$$

This gives us Equation 4, and thus concludes our proof. $\square$

Though it may seem counter-intuitive to create too many crossings and then unravel some of them, that is sometimes the most efficient way to represent the Torified Rational tangle with sticks. For example, a five-strand Torified Rational tangle with 24 σ moves could be made from the supercoil shown in Figure 17, and then subtracting a σ move. Since we are subtracting a single σ move, we can represent it without adding in any more sticks (see Figure 15). Therefore, we can create this tangle using only 25 sticks. Whereas, if we were to try and construct this tangle without undoing crossings, it would require 43 sticks.

Corollary 2. *The bound for stick number of a Torified Rational tangle, T, given by Equation 4 is always at least as good as the bound in Claim 2: $s_T(L) \leq 2(|x|-1)+n$.*

Proof. Consider the first case of Equation 4, $s_T \leq (3+2k)n+2(y-1)$. We want to identify when $(3+2k)n+2(y-1) \leq 2(x-1)+n$. Using the fact that $x = 3kn+2n+y$, we see that:

$$\begin{aligned}(3+2k)n+2(y-1) &\leq 2(x-1)+n\\ 3n+2nk+2y-2 &\leq 2x-2+n\\ 2n+2kn+2y &\leq 2x\\ n+kn+y &\leq x\\ n+kn+y &\leq 3kn+2n+y\\ 0 &\leq 2kn+n,\end{aligned}$$

which is always the case. Therefore the first case of Equation 4 is always at least as good as the bound in Claim 2. Furthermore, since the second case of Equation 4 is only used when it gives a better bound on stick number than the first, when we use it, it must also be at least as good as $2(|x|-1)+n$. $\square$

Claim 4. *A Torified Rational tangle can always be constructed such that it is entirely contained within a twist-box. In other words, each strand will only intersect the twist-box at its exit and entry points, which are on opposite faces of the box.*

Proof. Let T be a Torified Rational tangle of n strands. Then either T is not supercoiled, or it is. The case where there is no supercoiling, as in Figure 16, is trivial. Clearly we can fit a tangle of this kind in a twist-box in the desired way. The other case, however, is less straight-forward.

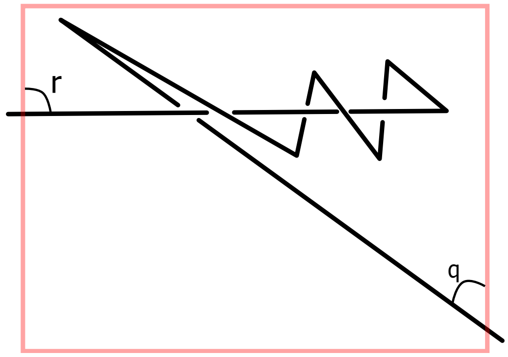


Figure 18: Supercoil inside a twist-box.

Consider the supercoil in Figure 18. We can imagine that the single strand which is supercoiled is actually a tube containing all n strands. Similarly, we notice that no matter how many twists we put in our supercoil, we can extend the ends so that it is inside a twist-box it in this way. Therefore, we can always fit T entirely in a twist-box, where each strand will only intersect the box at its exit and entry points, which are on opposite faces of the box. □

Corollary 3. *Let L be a three-box Torified Rational Link. If all of the twist-boxes of L have supercoils, then we can always attach the skeleton to the twist-boxes without adding any new sticks.*

Proof. We have more flexibility when the Torified Rational Tangles are supercoiled because we can pick the entry and exit angles of the strands.

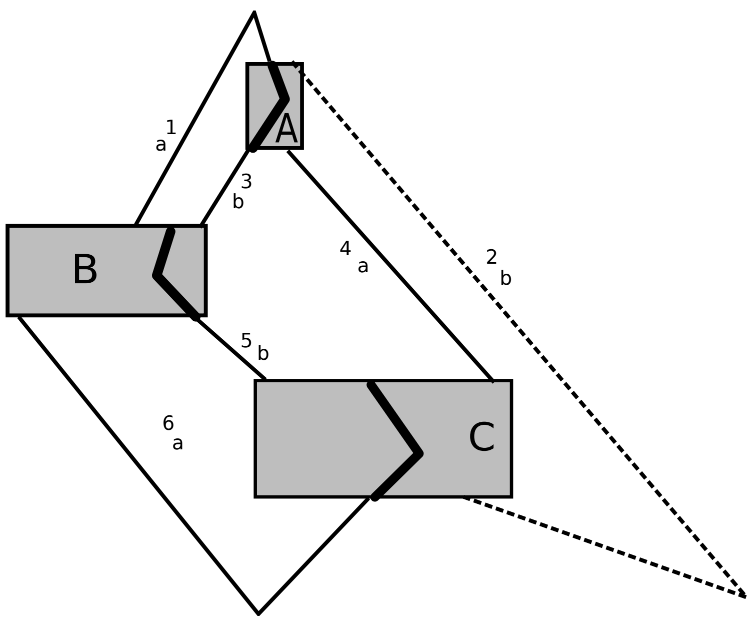


Figure 19: Sticks needed to build the skeleton of a Torified Rational Link with three twist-boxes, when all twist-boxes contain a supercoiled tangle.

Consider Figure 19. The thick black lines inside the twist-boxes show the entry and exit angles of the tangle within. We see that each portion of the skeleton contributes the following number of sticks:

Portion of Skeleton	Sticks Contributed
1	0
2	2b
3	-b
4	0
5	-b
6	0

The total maximum stick contribution of the skeleton is the sum of the contribution of each component of the skeleton, which is zero. Hence we can always attach the skeleton to the twist-boxes without adding any new sticks. □

Theorem 1. *Let L be a Torified Rational Link with three twist-boxes and n strands. Let the twist-boxes have x_1, x_2, and x_3 σ moves. Let the corresponding k_i be the largest integers such that $x_i = (2+3k_i)n + y_i$, and $y_i \geq 0$. Then define $s_i(L)$ by:*

$$s_i(L) = \begin{cases} (3+2k_i)n + 2(y_i - 1) & \text{if } y_i \leq 3n(k_i+1) \\ (5+2k_i)n + 2(x_i - y_i - 1) & \text{if } y_i > 3n(k_i+1). \end{cases} \tag{7}$$

Let $a = min\{w_1, w_2\}$. Then the number of sticks needed to build L is bounded by:

$$s(L) \leq 4n + a + \sum_{i=1}^{3} s_i(L). \tag{8}$$

Proof. We can build L by constructing its tangles and then clipping in its skeleton. Since we know from Lemma 1 that it takes at most $4n + a$ additional sticks to make the skeleton, the maximum number of sticks we need is the number needed to create each tangle, which we get from Lemma 2, plus $4n + a$. □

Corollary 4. *Let L be a Torified Rational Link with three twist-boxes and n strands. Let the twist-boxes have x_1, x_2, and x_3 σ moves. Suppose each twist-box contains a tangle that is supercoiled (i.e. $k_i \geq 0$, for all i). Let the corresponding k_i be the largest integers such that $x_i = (2+3k_i)n + y_i$, and $y_i \geq 0$. Then define $s_i(L)$ as in Equation 7. Then, the number of sticks needed to build L is bounded by:*

$$s(L) \leq \sum_{i=1}^{3} s_i(L). \tag{9}$$

Proof. This follows directly from Theorem 1 and Corollary 3, which saves us $4n + a$ sticks whenever all three twist-boxes L contain supercoiled tangles. □

Comparative bound on stick number

To compare our bound on stick number to that of Huh and Oh, we use Gruber's bound for $c(L)$ (Equation 1). The more complicated our Torified Rational Links are, the better our bound on stick number should be.

Often, our bound is better than that of Huh and Oh. For example, the HOMFLY polynomial of the link in Figure 20, arbitrarily oriented, yields $M = 18$, $E = -2$, and $e = -24$. This means that $c(L) \geq 29$, and therefore Huh and Oh's bound on stick number is, at best, $s(L) \leq 45$. However, using our bound (Equation 9), we get $s(L) \leq 35$, which is significantly better.

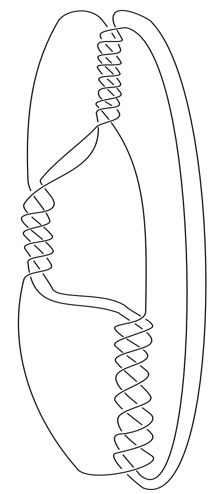


Figure 20: Torified Rational Link.

Links with n twist-boxes

Now we will consider the stick number of these more complicated Torified Rational Links. Fortunately, the construction of the tangles is almost the same. The only difference is that, unlike in the 3-box case, the twist boxes do not always have the same box strand number (see Figure 21).

Definition 5. *Each twist-box of a Torified Rational Link has a* ***box strand number****, n_b, which is the number of strands going into (or, equivalently, coming out of) the twist-box.*

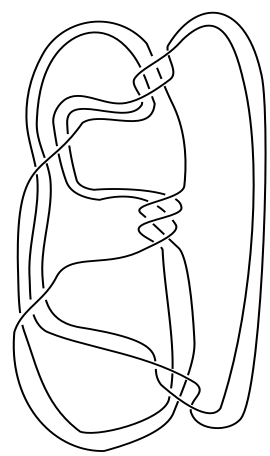
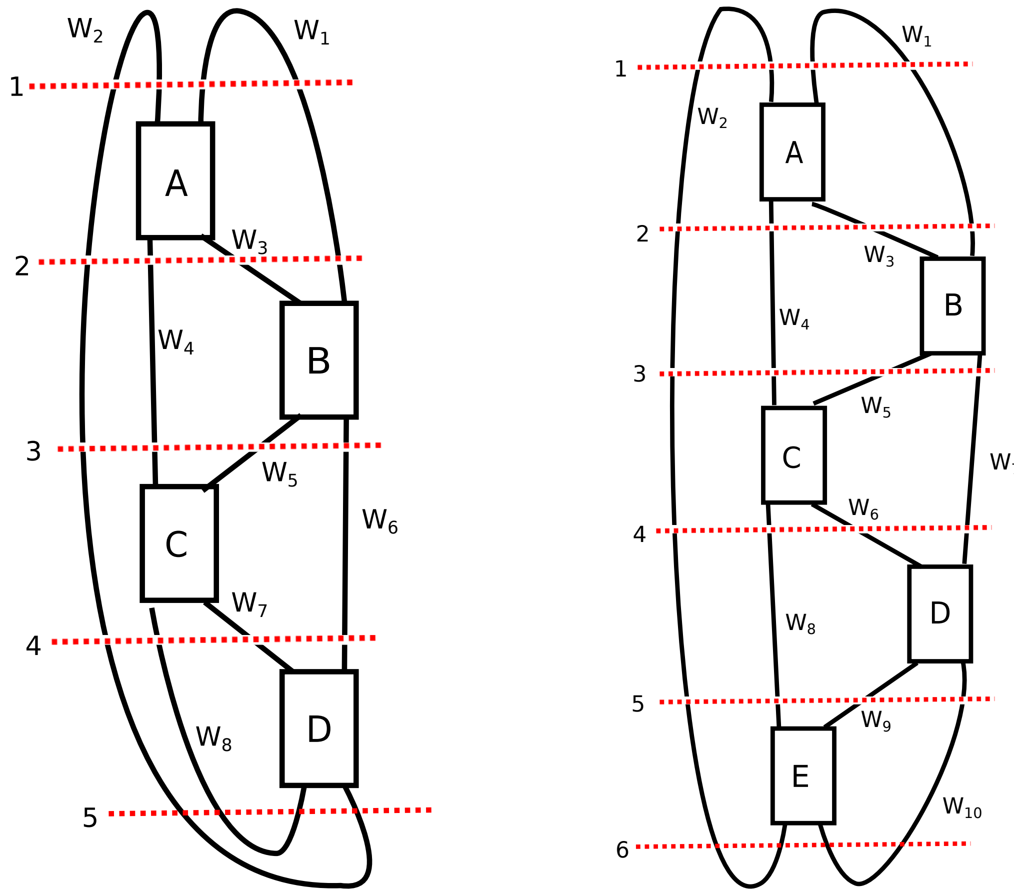


Figure 21: A Torified Rational Link whose twist-boxes do not all have the same box strand number.

Claim 5. *Let L be a Torified Rational Link with m twist-boxes. Let the w_i be the weights of the strands, numbered in the order shown in Figure 22. Then we will always have that $w_1 = w_{2m}$.*

Proof. Consider Figure 22, which shows a Torified Rational Link with an odd number of twist-boxes and one with an even number of twist-boxes. We know that the sum of the weights of the



(a) Even number of twist-boxes. (b) Odd number of twist-boxes

Figure 22: Symmetries and properties of Torified Rational Links.

two clusters of strands going into any box must be the same as the sum of the weights coming out of that box.

We want to show that the red lines in Figure 22 (a) intersect the same total number of strands. We know that $w_1 + w_2 = w_3 + w_4$. Therefore $w_1 + w_2 + w_1 + w_2 = w_1 + w_2 + w_3 + w_4$, and so lines 1 and 2 intersect the same total number of strands. Now, from box B we see that $w_1 + w_3 = w_5 + w_6$, so lines 2 and 3 intersect the same number of strands as well, since line 2 intersects $w_2 + w_4 + w_3 + w_1$ strands, and line 3 intersects $w_2 + w_4 + w_5 + w_6$ strands. Similarly, the relationships between the weights of the strands going through boxes C and D indicate that lines 3, 4, and 5 intersect the same number of strands. This tells us that lines 1 and 5 intersect the same number of strands. Therefore, $w_1 + w_2 + w_1 + w_2 {=} w_2 + w_8 + w_8 + w_2$, and hence $w_1 = w_8$. In other words, in this case, $w_1 = w_{2m}$. Notice that this pattern holds for any Torified Rational Link with an even number of twist-boxes, since each new box will add another relation on the w_i.

The odd case is essentially the same (See Figure 22 (b)). So we have that $w_1 = w_{2m}$ for all m. $\square$

Claim 6. *For any link, L, with m twist-boxes, $n_1 = n_m$.*

Proof. This follows directly from Claim 5, since the weights of the strands going out of box m (the last box) must equal the weights of the strands going into box 1 (the first box). □

Claim 7. *Let a_T be any Torified Rational tangle of a Torified Rational Link, L, whose first (top) twist-box has $n_1 = \kappa$. Then $n_T \leq 2\kappa - 2$.*

Proof. Again, this statement follows from the fact that the red lines in Figure 22 all intersect the same number of strands. Each red line intersects four of the w_i, at least two of which come from a single twist-box. The first red line intersects $2(w_1 + w_2)$ strands, and $w_1 + w_2 = \kappa$, so each line intersects 2κ strands. Consider the red line directly above a_T. It intersects the two w_i going into a_T and two other strands. We know that the sum of the two w_i going into a_T is equal to n_T, and each $w_i \geq 1$, so the maximum that n_T can be is $2\kappa - 2$. □

In Figure 22, notice that some of the collections of strands, w_i, intersect with one red line, and others intersect with two or more. We notice that w_1 and w_{2m} intersect twice with red lines, and w_2 intersects with each red line at least once. Excluding those three collections of strands, only the vertical inner w_i intersect with red lines twice.

Definition 6. *Define the **pair strands**, w_{l_i}, to be those strands which are intersected more than once by red lines, excluding w_1, w_2, and w_{2m}.*

Lemma 3. *Let L be a Torified Rational Link with m twist-boxes. Let $w_1 + w_2 = \kappa$, and let $\lambda = \sum_{i=1}^{m-2} w_{l_i}$. Then the number of sticks needed to construct the skeleton of L is bounded by:*

$$s_s(L) \leq m(2\kappa - w_2) - \lambda + w_2. \tag{10}$$

Proof. Consider Figure 23. By the same reasoning as for the case with only three twist-boxes (see Lemma 1), we know that this structure can exist in $\mathbb{R}^3$. Therefore, we want to use one stick for each strand in all clusters of strands except w_1, w_2, and w_{2m}. We know that the dotted symmetry lines shown in Figure 22 intersect exactly 2κ strands each. This means that lines $2, 3, ..., m$ intersect a total of $2\kappa(m-1)$ strands of the skeleton of L. We know that each of the w_i are intersected at least once by this collection of lines. The ones that are intersected more than once are the w_{l_i}, which are each intersected twice, and w_2, which is intersected $m-1$ times. So the total number of strands in the diagram is $\sum_{i=1}^{2m} w_i = 2\kappa(m-1) - (m-2)w_2 - \lambda$.

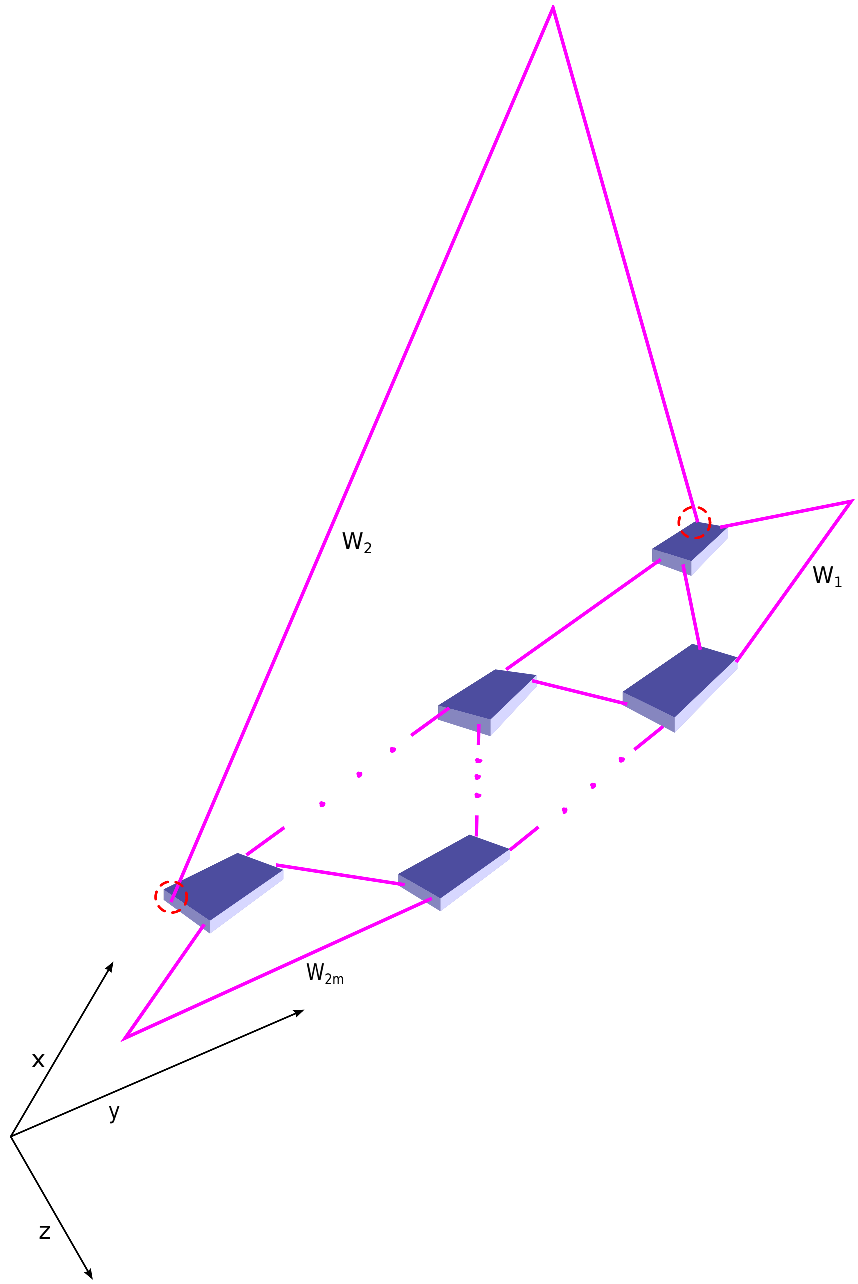


Figure 23: Skeleton of a Torified Rational Link with m twist-boxes made from sticks.

However, for our stick number, we need to add in $w_2 + w_1 + w_{2m}$ more sticks. Hence we have that $s_s(L) \leq 2\kappa(m-1) - (m-2)w_2 - \lambda + w_2 + w_1 + w_{2m}$. We know that $w_1 + w_2 = w_{2m} + w_2 = \kappa$. Therefore, $s_s(L) \leq m(2\kappa - w_2) - \lambda + w_2$. □

Now that we have a bound on the stick number of the skeleton of these Torified Rational Links, we need to find how many sticks are necessary to construct the tangles within each twist-box.

Theorem 2. *Let L be a Torified Rational Link with m twist-boxes containing tangles $T_1, T_2, ..., T_m$. Suppose each T_i has x_i σ moves and has box strand number n_i. Choose k_i to be the largest integer such that $x_i = (2+3k_i)n_i + y_i$, and $y \geq 0$. Let $w_1 + w_2 = \kappa$, and let $\lambda = \sum_{i=1}^{m-2} w_{l_i}$. Define $s_i(L)$ as in Equation 7. Then the stick number of L is bounded by:*

$$s(L) \leq m(2\kappa - w_2) - \lambda + w_2 + \sum_{i=1}^{m} s_i(L). \tag{11}$$

Proof. This follows from Lemma 2 and Lemma 3, since $s(L) \leq \sum_{i=1}^{m} s_i(L) + s_s(L)$. In other words, the stick number of a Torified Rational Link cannot be greater than the number of sticks needed to build its skeleton plus those needed to construct its tangles. □

Open Questions

1. How could we generally prove that our bound on stick number is better than $\frac{3}{2}(c(L)+1)$ using the HOMFLY polynomial?
2. In this paper, we replaced the integral tangles of rational links with Torified Rational ones. What would happen if we replaced them with other types of tangles?
3. A Torified Rational Link diagram with n twist-boxes looks like it should live on a genus n handlebody. It would be interesting to look at different covers of these links and see if we can find invariants based on the minimal genus handlebody it can inhabit.
4. There are many ways to approach identifying and studying a new class of links. What can other link invariants tell us about Torified Rational Links?
5. If a link can be represented as a Torified Rational Link, then will there exist a minimal crossing representation of that link that is also a Torified Rational Link?
6. If a Torified Rational Link, L, can be represented by a diagram, D, then will there always exist other Torified Rational representations of that link? If so, how many?

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