Linear Dependence and Hermitian Geometric Realizations of Canonical Algebraic Curvature Tensors

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Abstract

We bring the geometric nature of curvature tensors into the linear algebraic setting by studying inner product spaces equipped with an algebraic curvature tensor. Any point on a smooth Riemannian manifold gives rises to such a space in a canonical way with the tangent space equipped with restricted metric and Riemann curvature tensor. By studying the algebraic properties of these spaces, valuable geometric insight may be gained. It is known every algebraic curvature tensor can be written as a linear combination of canonical algebraic curvature tensors built from self-adjoint or skew-adjoint endomorphisms. We study the linear dependence of such canonical tensors with emphasis on the skew-adjoint setting. We then study the criteria for these tensors to be geometrically realizable on a Hermitian manifold.

1 Algebraic Curvature Tensors

Let V be an n dimensional real vector space and let $V^* := \{\varphi : V \to \mathbb{R} \,|\, \varphi \text{ linear}\}$ be the dual space of V. An *algebraic curvature tensor* is and element $R \in \otimes^4 V^*$ so that for all $x, y, z, w \in V$:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$

(3) is known as the first Bianchi identity. The set of all algebraic curvature tensors on V is itself a vector space which we denote by $\mathcal{A}(V)$.

1.1 Linear Algebra

Let $(V, (\cdot, \cdot))$ be a (positive definite) inner product space. Given a linear map $A : V \to V$ we define the *adjoint* map $A^* : V \to V$ by the equation

$$(Ax, y) = (x, A^*y) \qquad \forall x, y \in V.$$

In the case that we have $A^* = A$ then we say that A is *self-adjoint* or *symmetric*. If $A^* = -A$ then we say that A is *skew-adjoint* or *skew-symmetric*.

Given a collection $\mathcal{F} := \{A_1, \ldots, A_m\}$ of linear maps on V, we say that the collection is *simultaneously diagonalizable* if there exists an orthogonal linear transformation U so that U^*A_jU is diagonal for each $j = 1 \ldots m$. It is well known that this condition is equivalent to saying that collection $\mathcal{F}$ is

a commuting family and that each A_j is diagonalizable. The Spectral Theorem for normal operators (operators that commute with their adjoint) states that any normal operator N is orthogonally diagonalizable over $\mathbb{C}$. Precisely this means there exists a unitary operator U so that U^*NU is diagonal. Normal operators in general are *not* diagonalizable over $\mathbb{R}$ but in the case of skew-adjoint transformations we can obtain a block diagonal form, motivating the following definition. If $A : V \to V$ is a skew-adjoint linear map, then there exists a basis for V,
$\mathcal{B} := \{e_1, f_1, \ldots, e_n, f_n, h_1, \ldots, h_t\}$ so that $\ker(A) = \operatorname{span}\{h_1, \ldots, h_t\}$ and

$$(Af_j, e_j) = \lambda_j \neq 0 \quad \text{and} \quad (Ae_j, e_k) = (Af_j, f_k) = (Af_j, e_k) = 0 \quad \forall\ j \neq k.$$

So with respect to the basis $\mathcal{B}$ the matrix of A takes the form

$$A = \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \lambda_n & & & \\ & & & -\lambda_n & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \qquad \lambda_j \neq 0 \in \mathbb{R}$$

We then have $\operatorname{rank}(A) = 2n$ and so the rank must be even. We say that a collection $\mathcal{F} := \{A_1, \ldots, A_m\}$ of skew-adjoint linear maps are *simultaneously block diagonalizable* if there is a basis for V that puts A_j is the form above for each j. We note that if we are block diagonalizing a single matrix A, without loss of generality we can assume up to a permutation of coordinates that $\lambda_j > 0$, $j = 1, \ldots, n$. From this point on we clarify that we refer to inner product spaces as *positive definite* inner product spaces.

1.2 Canonical Algebraic Curvature Tensors

If A is any linear map on V then we define two elements in $\otimes^4 V^*$:

$$R_A^S(x, y, z, w) := (Ax, w)(Ay, z) - (Ax, z)(Ay, w),$$

and

$$R_A^\Lambda(x, y, z, w) := (Ax, w)(Ay, z) - (Ax, z)(Ay, w) - 2(Ax, y)(Az, w).$$

It is easy to see that if A is *self-adjoint* then R_A^S is an algebraic curvature tensor. Similarly if A is *skew-adjoint* then R_A^Λ is an algebraic curvature tensor as well. These are known as the *canonical algebraic curvature tensors*. It has been shown by Fiedler and Gilkey [2] that

$$\mathcal{A}(V) = \operatorname{span}\{R_A^S \mid A^* = A\} = \operatorname{span}\{R_A^\Lambda \mid A^* = -A\}.$$

Thus a study of these spanning sets provide insight to the space of all algebraic curvature tensors. We begin our study with some elementary algebraic properties and identities. We will adopt the convention for the rest of the paper that unless otherwise stated if an expression involves a canonical curvature tensor written as R_A with the superscript omitted, it is to be understood that the expression is true for both the self-adjoint and skew-adjoint constructions.

Lemma 1.1. *If A, B are linear maps on V and $\alpha \in \mathbb{R}$ then*

1. $R_{\alpha A} = \alpha^2 R_A$ *and* $\alpha R_A = \operatorname{sign}(\alpha) R_{\sqrt{|\alpha|}A}$.

2. $R_{A+B} + R_{A-B} = 2R_A + 2R_B$.

Proof. Immediate computation. □

Assertion (2) is the *Polarization Identity.* We remark that the Polarization Identity holds for any maps A, B. That is, viewed as elements of $\otimes^4 V^*$ the equation still is satisfied in both the self-adjoint and skew-adjoint constructions. It was shown in by A. Diaz and C. Dunn [1] that if $R_A^S \in \mathcal{A}(V)$ then $R_{A^*}^S \in \mathcal{A}(V)$ and $R_A^S = R_{A^*}^S$, we show that the same is true in the skew-adjoint case.

Lemma 1.2. *If $R_A^\Lambda \in \mathcal{A}(V)$ then $R_{A^*}^\Lambda \in \mathcal{A}(V)$ and $R_A^\Lambda = R_{A^*}^\Lambda$.*

Proof.

$$\begin{aligned} R_{A^*}^\Lambda(x,y,z,w) &= (A^*x,w)(A^*y,z) - (A^*x,z)(A^*y,w) - 2(A^*x,y)(A^*z,w) \\ &= (x,Aw)(y,Az) - (x,Az)(y,Aw) - 2(x,Ay)(z,Aw) \\ &= R_A^\Lambda(w,z,y,x) \\ &= R_A^\Lambda(x,y,z,w) \end{aligned}$$

□

Lemma 1.3. *If A is a self-adjoint linear map on V and we have that $R_A^\Lambda \in \mathcal{A}(V)$, then $A = 0$.*

Proof. Consider $R := R_A^\Lambda - R_A^S$. Then $R \in \mathcal{A}(V)$ so we have for $x, y, z, w \in V$

$$\begin{aligned} R(y,x,z,w) &= -2(Ay,x)(Az,w) \\ &= -2(Ax,y)(Az,w) \\ &= R(x,y,z,w) \\ &= -R(y,x,z,w) \end{aligned}$$

So we have $R = 0$. Therefore we see that

$$R(x,y,x,y) = -2(Ax,y)^2 = 0 \qquad \forall x, y \in V,$$

hence $A = 0$. □

In the self-adjoint case A. Diaz and C. Dunn [1] have shown that if $\text{Rank}(A) \geq 3$ then R_A^S is an algebraic curvature tensor if and only if A is self-adjoint. Working with S. Schmidt [9] we considered the skew-adjoint case and discovered the same to be true without any rank restriction.

Theorem 1.1. *$R_A^\Lambda \in \mathcal{A}(V)$ if and only if $A^* = -A$.*

Proof. One direction follows by a previous result in [2]. Assume now $R_A^\Lambda \in \mathcal{A}(V)$ then by the Polarization Identity and Lemma 1.2 above we have

$$R_{A+A^*}^\Lambda = 4R_A^\Lambda - R_{A-A^*}^\Lambda.$$

So it follows that $R_{A+A^*}^\Lambda$ is an algebraic curvature tensor because $(A - A^*)^* = -(A - A^*)$. Therefore by Lemma 1.3 we have $A + A^* = 0$ or $A^* = -A$. □

2 Linear Dependence

The canonical algebraic curvature tensors give rise to two distinct infinite spanning sets of $\mathcal{A}(V)$. Hence there is a lot of redundancy, leading to questions regrading linear dependence. Many different authors have addressed these questions but there are still many unanswered. The questions involving sets of 2 tensors have been completely solved. Questions involving sets of 3 tensors remains open except in the strictly self-adjoint case due to the work by A. Diaz and C. Dunn in [1]. B.K. Lovell [5] began answering these questions in the strictly skew-adjoint setting, providing motavation for further study. We look to expand on Lovell's results and to generalize basic ideas to sets of n algebraic curvature tensors in the skew-adjoint setting.

2.1 Setting up the Problem

The study of linear dependence of canonical algebraic curvature tensors is the analysis of the equation

$$\sum_{j=1}^{n} \alpha_j R_{A_j} = 0, \qquad \alpha_j \in \mathbb{R}.$$

for some collection of self-adjoint or skew-adjoint linear maps A_j on V. By Lemma 1.1 above we can reduce the above equation by setting $\epsilon_j := \operatorname{sign}(\alpha_j)$ and $B_j := \sqrt{|\alpha_j|}A_j$ obtaining

$$\sum_{j=1}^{n} \epsilon_j R_{B_j} = 0, \qquad \epsilon_j \in \{-1, 1\}.$$

Thus without loss of generality it suffices to study the sums and differences of canonical algebraic curvature tensors. The following lemma due to Gilkey [3] will prove very useful later in the paper. We state it here and refer to [3] for proof.

Lemma 2.1 (Gilkey [3])**.** *We have*

1. *If $A_1, A_2 : V \to V$ are self-adjoint linear maps with* $\operatorname{Rank}(A_1) \geq 3$ *and $R^S_{A_1} = R^S_{A_2}$ then $A_1 = \pm A_2$.*
2. *If $A_1, A_2 : V \to V$ are skew-adjoint linear maps and $R^\Lambda_{A_1} = R^\Lambda_{A_2}$ then $A_1 = \pm A_2$.*

Using this result, other authors have completely answered the linear dependence question when two curvature tensors are involved. We state the summarized result here due to A. Diaz, C. Dunn [1] and Treadway [4].

Lemma 2.2 (Diaz, Dunn [1] and Treadway [4])**.** *We have*

1. *Suppose $A : V \to V$ is a self-adjoint linear map with* $\operatorname{Rank}(A) \geq 3$ *then there does not exist a self-adjoint B so that $R^S_A = -R^S_B$.*
2. *Suppose $A : V \to V$ is a non-zero skew-adjoint linear map with* $\operatorname{Rank}(A) \geq 4$*, then there does not exist a skew-adjoint B so that $R^\Lambda_A = -R^\Lambda_B$.*

A careful inspection of the proof of Assertion (2) will show that in fact the result is true in the rank 2 case as well. The authors of [1] studied the linear dependence question in setting of sets of 3 canonical algebraic curvature tensors built from self-adjoint maps. In the following section we wish to prove the analogous result in the skew-adjoint setting.

2.2 Linear Dependence in the Skew-Adjoint Setting

In [1], the main result regarded the linear dependence of 3 canonical algebraic curvature tensors and provided a spectral criteria on the endomorphisms involved to ensure there would be a dependence relationship. These spectral requirements were very rigid, and we shall show that in the skew-adjoint situation the relationship is even more rigid but remarkably straight forward.

Lemma 2.3. *Let A, B, C are non-zero skew-adjoint linear maps on an inner product space $(V, (\cdot, \cdot))$.*

1. *If $R^\Lambda_A = R^\Lambda_B + R^\Lambda_C$, then $B = \alpha C$ for some $\alpha \in \mathbb{R}$.*
2. *$R^\Lambda_A + R^\Lambda_B + R^\Lambda_C \neq 0$*

Proof. Consider the equations,

$$R^\Lambda_A = \epsilon(R^\Lambda_B + R^\Lambda_C) \qquad \epsilon \in \{-1, 1\}$$

Since A is skew-symmetric and non-zero we can obtain a orthonormal basis for $(V, (\cdot,\cdot))$ $\mathcal{B}:=\{e_1, f_1, \ldots, e_n, f_n, h_1, \ldots, h_t\}$ so that $\ker(A) = \text{span}\{h_1, \ldots, h_t\}$ and

$$(Ae_j, e_i) = (Af_j, f_i) = 0 \;\text{ and }\; (Af_j, e_j) := a_j > 0 \;\; i, j = 1, \ldots, n$$

Note the assumption that $A \neq 0$ and the fact that $\text{Rank}(A)$ is even implies $n \geq 1$. The matrix of A in this basis takes the form

$$A = \begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & a_n & & & \\ & & & -a_n & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix} \qquad a_j > 0$$

First we show that $\epsilon \neq -1$ for if it did then consider

$$R^\Lambda_A(e_j, f_j, f_j, e_j) = 3a_j^2 > 0$$

but

$$-(R^\Lambda_B + R^\Lambda_C)(e_j, f_j, f_j, e_j) = -3((Bf_j, e_j)^2 + (Cf_j, e_j)^2) \leq 0.$$

So $\epsilon \neq -1$ and thus Assertion (2) is established.

To see Assertion (1) we will first show that B and C must also be block diagonal. If $j \neq k$ then consider

$$0 = R^\Lambda_A(e_j, e_k, e_k, e_j) = (R^\Lambda_B + R^\Lambda_C)(e_j, e_k, e_k, e_j) = 3((Be_k, e_j)^2 + (Ce_k, e_j)^2),$$

therefore $(Be_k, e_j) = (Ce_k, e_j) = 0$ for $j \neq k$. A similar calculation will show that $(Bf_k, f_j) = (Bf_k, e_j) = (Cf_k, f_j) = (Cf_k, e_j) = 0$ for $j \neq k$ and also that the kernels coincide. Therefore B and C are block diagonal in this basis as well.

To simplify notation let

$$b_j := (Bf_j, e_j) \;\text{ and }\; c_j := (Cf_j, e_j)$$

Now we compute

$$2a_ja_k = R^\Lambda_A(e_j, f_j, f_k, e_k) = (R^\Lambda_B + R^\Lambda_C)(e_j, f_j, f_k, e_k) = 2(b_jb_k + c_jc_k),$$

so we are lead to the equations

$$a_ja_k = b_jb_k + c_jc_k \qquad \forall\, j, k.$$

In particular putting $k = j$ we have $a_j^2 = b_j^2 + c_j^2$. Now fix $k = 1$, by squaring the equation above we obtain

$$a_j^2a_1^2 = (b_jb_1 + c_jc_1)^2 = b_j^2b_1^2 + 2b_jb_1c_jc_1 + c_j^2c_1^2,$$

since $a_j^2 = b_j^2 + c_j^2$ we have

$$(b_j^2 + c_j^2)a_1^2 - b_j^2b_1^2 - 2b_jb_1c_jc_1 - c_j^2c_1^2 = 0.$$

So

$$\begin{aligned} b_j^2(a_1^2 - b_1^2) + c_j^2(a_1^2 - c_1^2) - 2b_jb_1c_jc_1 &= b_j^2c_1^2 + c_j^2b_1^2 - 2b_jb_1c_jc_1 \\ &= (b_jc_1 - c_jb_1)^2 \\ &= 0. \end{aligned}$$

Thus we conclude that $b_jc_1 = c_jb_1$, and since $a_1 \neq 0$ we have either $b_1 \neq 0$ or $c_1 \neq 0$. Without loss of generality we may assume $c_1 \neq 0$, hence we are lead to $b_j = \frac{b_1}{c_1}c_j$ for $j = 1, \ldots, n$. It then follows immediately that $B = \alpha C$. $\square$

Theorem 2.1. *If A_1, A_2, A_3 are non-zero skew-adjoint linear maps on an inner product space $(V, (\cdot,\cdot))$. Then the set $\{R^\Lambda_{A_1}, R^\Lambda_{A_2}, R^\Lambda_{A_3}\}$ is linearly dependent if and only if $A_j = \alpha A_i$ for some $j \neq i$ and $\alpha \in \mathbb{R}$.*

Proof. First assume we have linear dependence,

$$\alpha_1 R^\Lambda_{A_1} + \alpha_2 R^\Lambda_{A_2} + \alpha_3 R^\Lambda_{A_3} = 0.$$

For some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ not all zero. If two coefficients are zero then by Lemma 2.1 A_j must be the zero map for some j, reaching a contradiction.

If we have $\alpha_j = 0$ for some j, then without loss of generality, after relabeling we can assume $\alpha_3 = 0$. Putting $\epsilon = \text{sign}(\frac{\alpha_2}{\alpha_1})$, $A := \sqrt{|\alpha_1|}A_1$ and $B := \sqrt{|\alpha_2|}A_2$ then in accordance to the previous remarks we use Lemma 1.1 to obtain the new tensor equation,

$$R^\Lambda_A = \epsilon R^\Lambda_B.$$

If $\epsilon = -1$ then we apply Lemma 2.2 and conclude $A = 0$ obtaining a contradiction. Thus $\epsilon = 1$ and we apply Lemma 2.1 to see that $A = \pm B$, hence $A_1 = \alpha A_2$ for some $\alpha \in \mathbb{R}$.

Now assume $\alpha_j \neq 0$ for each $j = 1, 2, 3$. Similar to above put $\epsilon_j = \text{sign}(\frac{\alpha_j}{\alpha_1})$, $A := \sqrt{|\alpha_1|}A_1$, $B := \sqrt{|\alpha_2|}A_2$ and $C := \sqrt{|\alpha_3|}A_3$. Then by symmetry in A_1, A_2, and A_3 it suffices to consider the equations,

$$R^\Lambda_A = \epsilon(R^\Lambda_B + R^\Lambda_C) \qquad \epsilon \in \{-1, 1\}.$$

In view of Lemma 2.3 we have $\epsilon = 1$ and $B = \beta C$, hence $A_2 = \alpha A_3$ with $\beta, \alpha \in \mathbb{R}$. In fact more can be said, it then follows from Lemma 2.1 that we have $A_1 = \gamma A_2$, $\gamma \in \mathbb{R}$ and so all of A_1, A_2, A_3 are real multiples of one another. Since the converse is trivial the proof is complete. $\square$

We have the following immediate corollary.

Corollary 2.1. *If A_1, A_2, A_3 are linearly independent non-zero skew-adjoint linear maps on an inner product space $(V, (\cdot,\cdot))$, then the corresponding algebraic curvature tensors $R^\Lambda_{A_1}, R^\Lambda_{A_2}, R^\Lambda_{A_3}$ are linearly independent.*

Some ideas from the proof of Theorem 3 can be generalized to sets of n algebraic curvature tensors. Precisely, we are able to make two statements regarding the endomorphisms, solely derived from the structure of a tensor equation. Notice below the expressions involved only concern *adding* the tensors $R^\Lambda_{A_j}$.

Lemma 2.4. *Let $A_1, \ldots, A_n : V \to V$ be skew-adjoint linear maps.*

1. *If $\sum_{j=1}^n R^\Lambda_{A_j} = 0$ then $A_1 = \cdots = A_n = 0$.*

2. *If $R^\Lambda_{A_1} = \sum_{j=2}^n R^\Lambda_{A_j}$ then $A_1, \ldots, A_n$ are simultaneously block diagonalizable.*

Proof. Choose a basis for $\ker(A_1)^\perp$ $\{e_1, f_1, \ldots, e_n, f_n\}$ that block diagonalizes A_1 with $Af_j = \lambda_j e_j$, $Ae_j = -\lambda_j f_j$, $\lambda_j \geq 0$. Re-writing the above expression and evaluating at (e_k, f_k, f_k, e_k)

$$\begin{aligned} R^\Lambda_{A_1}(e_k, f_k, f_k, e_k) &= -\textstyle\sum_{j=2}^n R^\Lambda_{A_j}(e_k, f_k, f_k, e_k) \\ &= -3\textstyle\sum_{j=2}^n (A_j f_k, e_k)^2 \\ &\leq 0. \end{aligned}$$

Since $R^{\Lambda}_{A_1}(e_k, f_k, f_k, e_k) = 3\lambda_k^2 \geq 0$ we conclude $\lambda_k = 0$ for each k and hence A_1 must be the zero map. Repeating the process to $A_2, A_3, \dots$ will show $A_j = 0$ for each j. This proves (1), to see (2) let $\{e_1, f_1, \dots, e_n, f_n\}$ be the basis that block diagonalized A_1 as above. Now consider for $k \neq l$

$$\begin{aligned} R^{\Lambda}_{A_1}(e_l, e_k, e_k, e_l) &= \textstyle\sum_{j=2}^n R^{\Lambda}_{A_j}(e_l, e_k, e_k, e_l) \\ &= 3\textstyle\sum_{j=2}^n (A_j e_k, e_l)^2 \\ &\geq 0 \end{aligned}$$

Since $R^{\Lambda}_{A_1}(e_l, e_k, e_k, e_l) = 0$ it follows that $(A_j e_k, e_l) = 0$ for each $k \neq l$. Replacing e_k, e_l in the above calculation with f_k, f_l or f_k, e_l will yield $(A_j f_k, f_l) = (A_j f_k, e_l) = 0$ for $k \neq l$. Therefore it follows that A_j are in block diagonal form with respect to the basis $\{e_1, f_1, \dots, e_n, f_n\}$ on $\ker(A_1)^{\perp}$. Letting $\{h_1, \dots, h_t\}$ be a basis for $\ker(A_1)$ we easily see by substituting once more above that all the kernels coincide.

□

Theorem 2.2. *Let $\Omega, A_1, \dots, A_n : V \to V$ be skew-adjoint linear maps with $\Omega \neq 0$. If*

$$R^{\Lambda}_{\Omega} = \sum_{j=1}^n R^{\Lambda}_{A_j}$$

then $\Omega, A_1, \dots, A_n$ are linearly dependent.

Proof. We follow in the footsteps of theorem 3. By Lemma 3 there exists a basis for V $\{e_1, f_1, \dots, e_n, f_n, h_1, \dots, h_t\}$ that block diagonalizes $\Omega, A_1, \dots, A_n$. We denote the non-zero entries of A_j as $a_j^{(k)} := (A_j f_k, e_k)$ and $\omega_k := (\Omega f_k, e_k)$ for Ω. We compute

$$\begin{aligned} 2\omega_l\omega_k = R^{\Lambda}_{\Omega}(e_l, f_l, f_k, e_k) &= \textstyle\sum_{j=1}^n R^{\Lambda}_{A_j}(e_l, f_l, f_k, e_k) \\ &= 2\textstyle\sum_{j=1}^n a_j^{(l)} a_j^{(k)}. \end{aligned}$$

Thus we are lead to the eigenvalue equations

$$\omega_l\omega_k = \sum_{j=1}^n a_j^{(l)} a_j^{(k)} \qquad \forall j, k = 1, \dots, n.$$

Since $\Omega \neq 0$ we can assume without loss of generality that $\omega_1 \neq 0$. Setting $l = 1$ in the eigenvalue equation above we obtain

$$\omega_k = \sum_{j=1}^n \frac{a_j^{(1)}}{\omega_1} a_j^{(k)} \qquad \forall j, k = 1, \dots, n$$

which shows

$$\Omega = \sum_{j=1}^n \frac{a_j^{(1)}}{\omega_1} A_j$$

as desired.

□

Thus Theorem 2.2 can be regarded as a generalization of Theorem 2.1. In the case with 3 tensors the only possible equation was of the form in Theorem 2.2. Furthermore the eigenvalue equation factored and reduced down to give a direct proportionality relationship between the endomorphisms involved.

3 Geometric Realization on Hermitian Manifolds

The purpose of studying algebraic curvature tensors is to bring a notion of geometry into an algebraic setting in hopes of gaining geometric insight. Thus algebraic curvature tensors are intended to represent geometric objects. A natural question is if every algebraic curvature tensor is geometrically realizable at a point on some manifold. That is to say given $R \in \mathcal{A}(V)$, is there a point p on some Riemannian manifold (M, g) so that the given algebraic curvature tensor R is isometric to the Riemann curvature tensor at the point p? Gilkey showed in [2] that the answer to this question is yes. An important subclass of Riemannian manifolds are Hermitian manifolds, which are the analog of Riemannian manifolds in complex geometry. It is not the case that every algebraic curvature tensor is geometrically realizable by a Hermitian manifold. Several authors in [6] characterized exactly when this is the case for an arbitrary $R \in \mathcal{A}(V)$. We use their result and focus on the canonical algebraic curvature tensors to examine the criteria on the endomorphisms used to construct them.

3.1 Complex Curvature Models

Following definitions in [6] we let $(V, (\cdot, \cdot))$ be a real inner product space of dimension $2n$. An *almost complex structure* on V is a linear map J that satisfies

$$J^2 = -I \quad \text{and} \quad J^*(\cdot, \cdot) = (\cdot, \cdot).$$

Where J^* denotes the pullback defined by $J^*(x, y) = (Jx, Jy)$. With an almost complex structure on a vector space there is a notion of a *complex line.* A *complex line* is a 2 dimensional subspace $\pi \subset V$ so that $J\pi \subset \pi$. If R is an algebraic curvature tensor on V then we say that $(V, (\cdot, \cdot), J, R)$ is a *complex curvature model.*

A *complex structure* on a manifold M is a tensor field $\mathcal{J}$ so that at each point $p \in M$, $\mathcal{J}_p$ is and almost complex structure on T_pM and there are local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on a neighborhood of p so that

$$\mathcal{J}\partial_{x_j} = \partial_{y_j} \qquad \mathcal{J}\partial_{y_j} = -\partial_{x_j}.$$

A *Hermitian manifold* $\mathcal{M}$ is a triple $\mathcal{M} := (M, g, \mathcal{J})$ where M is a $2n$ real-dimensional manifold, g a Riemannian metric and $\mathcal{J}$ a complex structure on M.

We say that a complex curvature model $(V, (\cdot, \cdot), J, A)$ is *geometrically realizable by a Hermitian manifold* if for some point p on a Hermitian manifold $\mathcal{M}$ there exists an isometry φ from V to TM satisfying

$$\varphi^*\mathcal{J}_p = J \qquad \varphi^* g_p = (\cdot, \cdot) \qquad \varphi^* R_p = A$$

where R_p is the Riemann curvature tensor of (M, g) at p.

3.2 The Gray Identity

It is known exactly when a complex curvature model is geometrically realizable on a *Hermitian* manifold. It was shown by several authors in [6] that this is the case if and only if the curvature tensor satisfies the *Gray Identity.* That is, if R is an algebraic curvature tensor then the Gray Identity states

$$\begin{aligned} R(x, y, z, w) + R(Jx, Jy, Jz, Jw) \quad = \quad & R(Jx, Jy, z, w) + R(x, y, Jz, Jw) \\ & + R(Jx, y, Jz, w) + R(x, Jy, z, Jw) \\ & + R(Jx, y, z, Jw) + R(x, Jy, Jz, w). \end{aligned}$$

We now aim to characterize when *canonical* algebraic curvature tensors satisfy the Gray Identity and hence are geometrically realizable by a Hermitian manifold. Gilkey [2] provided a partial result to this question which we expand on.

Lemma 3.1 (Gilkey [2])**.** *Let $A : V \to V$ be a linear map. Let $(V, (\cdot, \cdot))$ be an inner product space endowed an almost complex structure J. Let $A^* = \pm A$, put $R_A := R_A^S$ if $A^* = A$ and $R_A := R_A^\Lambda$ if $A^* = -A$. Then*

1. *If $JA = AJ$ then R_A satisfies the Gray Identity.*

2. *If $JA = -AJ$ and if $\mathrm{Rank}(A) \leq 2$ then R_A satisfies the Gray Identity.*

3. *If $JA = -AJ$ and if $\mathrm{Rank}(A) > 2$ then R_A violates the Gray Identity.*

The main result in Theorem 3.1 shows that the restrictions on skew-adjoint operators are more rigid than that of self-adjoint maps.

Lemma 3.2. *Let $A : V \to V$ be a self-adjoint linear map. Let $(V, (\cdot, \cdot))$ be an inner product space endowed an almost complex structure J. Then $\mathrm{Rank}(AJ - JA) \neq 1$.*

Proof. We argue by contradiction, put $B := AJ - JA$ and suppose $\mathrm{Rank}(B) = 1$. First see that B and J anti-commute

$$JB = JAJ + A = (JA - AJ)J = -(AJ - JA)J = -BJ.$$

Since $B^* = (AJ - JA)^* = -JA + AJ = B$, by the Spectral Theorem for self-adjoint operators we can find a vector $e_1 \in V$ so that $Be_1 = \lambda e_1$ with $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Now consider

$$BJe_1 = -JBe_1 = -J(\lambda e_1) = -\lambda Je_1.$$

Thus Je_1 is an eigenvector of B corresponding to the eigenvalue $-\lambda$. Since it was assumed that $\lambda \neq 0$, we conclude that B has two distinct eigenvalues and thus $\mathrm{Rank}(B) \geq 2$, which gives us our contradiction.

□

Lemma 3.3. *Let $(V, (\cdot, \cdot))$ be an inner product space endowed with an almost complex structure J. Let $A : V \to V$ be a linear map. Then*

1. *R^S_A satisfies the Gray Identity if and only if $R^S_{AJ-JA} = R^S_{A+JAJ}$.*

2. *R^Λ_A satisfies the Gray Identity if and only if $R^\Lambda_{AJ-JA} = R^\Lambda_{A+JAJ}$.*

Proof. We first prove the self-adjoint result $R^S_{AJ-JA} = R^S_{A+JAJ}$. Consider the right hand side of the Gray Identity,

$$\begin{aligned}
&R^S_A(Jx, Jy, z, w) + R^S_A(x, y, Jz, Jw) \\
&+R^S_A(Jx, y, Jz, w) + R^S_A(x, Jy, z, Jw) \\
&+R^S_A(Jx, y, z, Jw) + R^S_A(x, Jy, Jz, w)
\end{aligned}
\quad
\begin{aligned}
= \;& (AJx, w)(AJy, z) - (AJx, z)(AJy, w) \\
&+(Ax, Jw)(Ay, Jz) - (Ax, Jz)(Ay, Jw) \\
&+(AJx, w)(Ay, Jz) - (AJx, Jz)(Ay, w) \\
&+(Ax, Jw)(AJy, z) - (Ax, z)(AJy, Jw) \\
&+(AJx, Jw)(Ay, z) - (AJx, z)(Ay, Jw) \\
&+(Ax, w)(AJy, Jz) - (Ax, Jz)(AJy, w) \\
= \;& (AJx, w)\Big[(AJy, z) + (Ay, Jz)\Big] \\
&+(Ax, Jw)\Big[(Ay, Jz) + (AJy, z)\Big] \\
&-(AJx, z)\Big[(AJy, w) + (Ay, Jw)\Big] \\
&-(Ax, Jz)\Big[(Ay, Jw) + (AJy, w)\Big] + \Theta
\end{aligned}$$

Where $\Theta = (AJx, Jw)(Ay, z) + (Ax, w)(AJy, Jz) - (AJx, Jz)(Ay, w) - (Ax, z)(AJy, Jw)$. Let $B := AJ - JA$, then further simplification leads to

$$\begin{aligned}
&R^S_A(Jx, Jy, z, w) + R^S_A(x, y, Jz, Jw) \\
&+R^S_A(Jx, y, Jz, w) + R^S_A(x, Jy, z, Jw) \\
&+R^S_A(Jx, y, z, Jw) + R^S_A(x, Jy, Jz, w)
\end{aligned}
\quad
\begin{aligned}
= \;& (AJx, w)(By, z) + (Ax, Jw)(By, z) \\
&-(AJx, z)(By, w) - (Ax, Jz)(By, w) + \Theta \\
\\
= \;& (Bx, w)(By, z) - (Bx, z)(By, w) + \Theta \\
= \;& R^S_B(x, y, z, w) + \Theta
\end{aligned}$$

Now consider the left hand side of the Gray identity with Θ subtracted.

$$\begin{aligned}
R^S_A(x,y,z,w)+R^S_A(Jx,Jy,Jz,Jw)-\Theta &= (Ax,w)(Ay,z)-(Ax,z)(Ay,w)\\
&\quad +(AJx,Jw)(AJy,Jz)-(AJx,Jz)(AJy,Jw)\\
&\quad -(AJx,Jw)(Ay,z)-(Ax,w)(AJy,Jz)\\
&\quad +(AJx,Jz)(Ay,w)+(Ax,z)(AJy,Jw)\\
&= (Ax,w)\Big[(Ay,z)-(AJy,Jz)\Big]\\
&\quad -(AJx,Jw)\Big[(Ay,z)-(AJy,Jz)\Big]\\
&\quad -(Ax,z)\Big[(Ay,w)-(AJy,Jw)\Big]\\
&\quad +(AJx,Jz)\Big[(Ay,w)-(AJy,Jw)\Big]
\end{aligned}$$

Now let $C := A + JAJ$, then we simplify further to obtain

$$\begin{aligned}
R^S_A(x,y,z,w)+R^S_A(Jx,Jy,Jz,Jw)-\Theta &= (Ax,w)(Cy,z)-(AJx,Jw)(Cy,z)\\
&\quad -(Ax,z)(Cy,w)+(AJx,Jz)(Cy,w)\\
&= (Cy,z)(Cx,w)-(Cy,w)(Cx,z)\\
&= R^S_C(x,y,z,w).
\end{aligned}$$

And so the Gray identity reduces down to $R^S_{AJ-JA}=R^S_{A+JAJ}$. To show this holds in the skew-adjoint setting we define for convenience the tensor $\phi_A \in \otimes^4 V^*$ as the following

$$\phi_A(x,y,z,w) := (Ax,y)(Az,w)$$

Thus for any linear map A we have the tensor relationship $R^\Lambda_A = R^S_A - 2\phi_A$. To establish Assertion (1) we did not impose any restrictions on the map A, hence it suffices to show the tensor ϕ_A reduces the Gray Identity to $\phi_{AJ-JA}=\phi_{A+JAJ}$. Similarly as above we define $B := AJ - JA$ and $\tilde{\Theta} := \phi_A(Jx,Jy,z,w)+\phi_A(x,y,Jz,Jw)$. Plugging ϕ into the Gray Identity we first simplify the right hand side.

$$\begin{aligned}
\phi_A(Jx,Jy,z,w)+\phi_A(x,y,Jz,Jw) &= \phi_A(Jx,Jy,z,w)+\phi_A(x,y,Jz,Jw)\\
+\phi_A(Jx,y,Jz,w)+\phi_A(x,Jy,z,Jw) &\quad +(AJx,y)(AJz,w)+(Ax,Jy)(Az,Jw)\\
+\phi_A(Jx,y,z,Jw)+\phi_A(x,Jy,Jz,w) &\quad +(AJx,y)(Az,Jw)+(Ax,Jy)(AJz,w)\\
&= (AJx,y)\Big[(AJz,w)+(Az,Jw)\Big]\\
&\quad (Ax,Jy)\Big[(AJz,w)+(Az,Jw)\Big]+\tilde{\Theta}\\
&= (Bz,w)\Big[(AJx,y)+(Ax,Jy)\Big]+\tilde{\Theta}\\
&= \phi_B(x,y,z,w)+\tilde{\Theta}
\end{aligned}$$

Subtracting $\tilde{\Theta}$ from the left hand side and letting $C := A + JAJ$ be as above, we compute

$$\begin{aligned}
\phi_A(x,y,z,w)+\phi_A(Jx,Jy,Jz,Jw)-\tilde{\Theta} &= (Ax,y)(Az,w)+(AJx,Jy)(AJz,Jw)\\
&\quad -(AJx,Jy)(Az,w)-(Ax,y)(AJz,Jw)\\
&= (Ax,y)\Big[(Az,w)-(AJz,Jw)\Big]\\
&\quad +(AJx,Jy)\Big[(AJz,Jw)-(Az,w)\Big]\\
&= (Cz,w)\Big[(Ax,y)-(AJx,Jy)\Big]\\
&= (Cx,y)(Cz,w)\\
&= \phi_C(x,y,z,w).
\end{aligned}$$

So the Gray Identity also reduces down to the equation $R^{\Lambda}_{AJ-JA} = R^{\Lambda}_{A+JAJ}$ in the skew-adjoint case. $\square$

We remark that similar to the Polarization Identity, the Gray Identity reduces down to the forms above for any tensor $R_A \in \otimes^4 V^*$ as we did not assume that A was self-adjoint or skew-adjoint. Under this simplification we now use Lemma 3.2 to state necessary and sufficient conditions for a canonical algebraic curvature tensor to be geometrically realizable on a Hermitian manifold.

Theorem 3.1. *Let $(V, (\cdot,\cdot))$ be an inner product space endowed with an almost complex structure J. Let A be a linear map on V.*

1. *If $A^* = -A$ then the complex curvature model $(V, (\cdot,\cdot), J, R^{\Lambda}_A)$ is geometrically realizable on a Hermitian manifold if and only if*
$$AJ = JA.$$
2. *If $A^* = A$ then the complex curvature model $(V, (\cdot,\cdot), J, R^{S}_A)$ is geometrically realizable on a Hermitian manifold if and only if there exists a complex line π so that*
$$AJ\big|_{\pi^\perp} = JA\big|_{\pi^\perp}$$
i.e. A commutes with J on the orthogonal complement of some complex line.

Proof. Suppose $A^* = -A$. Let $B := AJ - JA$ and $C := JB = A + JAJ$ and assume we have $R^{\Lambda}_B = R^{\Lambda}_C$ (Note that B and C are skew-adjoint so these are indeed algebraic curvature tensors). By Lemma 2.1 we have that
$$B = \pm C = \pm JB,$$
but this equation is equivalent to
$$B(I \mp J) = 0.$$
Multiplying on the right by $(I \pm J)$ we obtain
$$\begin{aligned} B(I \mp J)(I \pm J) &= 0, \\ B(I^2 - J^2) &= 0, \\ 2B &= 0. \end{aligned}$$
Hence $B = 0$ and A and J must commute. On the other hand if A and J commute then $B = C = 0$ and the result holds. This establishes assertion (1).

Now suppose $A^* = A$. Then applying Lemma 2.1 again we have two cases, $\text{Rank}(B) \geq 3$ and $\text{Rank}(B) < 3$. If we have $\text{Rank}(B) \geq 3$ then we have $B = \pm C$. But as shown above this equation implies that $B = 0$, contradicting the fact that $\text{Rank}(B) \geq 3$. So we must have $\text{Rank}(B) < 3$. For now assume $\text{Rank}(B) = 2$, then put $\pi := \ker(B)^\perp$. So we have $\dim(\pi) = 2$. Since $B^* = B$ we have for any $y \in \ker(B)$
$$(Bx, y) = (x, By) = 0 \qquad \forall x \in V.$$
In particular this holds for $x \in \pi$ and therefore the restriction $B\big|_\pi : \pi \to \pi$ makes sense. Also since $\text{Rank}(B) = 2$ it follows that $B\big|_\pi$ is invertible and that $B\big|_\pi^* = B\big|_\pi$. So by the Spectral Theorem for self-adjoint maps we can find an orthonormal basis for π, $\{e_1, e_2\}$ so that
$$Be_1 = \lambda_1 e_1 \text{ and } Be_2 = \lambda_2 e_2 \qquad \lambda_1, \lambda_2 \neq 0.$$
We now wish to show that π is a complex line. Since J and B anti-commute consider
$$BJe_1 = -JBe_1 = -J(\lambda_1 e_1) = -\lambda_1 Je_1,$$
i.e., Je_1 is an eigenvector of B corresponding to the eigenvalue $-\lambda_1$. Since $\text{spec}(B) = \{\lambda_1, \lambda_2\}$ we must have $\lambda_2 = -\lambda_1$ and moreover that $Je_1 = \pm e_2$ since Je_1 is a unit vector in the eigenspace

$\ker(B+\lambda_1 I) = \mathrm{span}\{e_2\}$. A similar calculation will show that $Je_2 = \pm e_1$ and therefore in particular we have $J\pi \subset \pi$, hence π is a complex line and by definition of π we have

$$B\big|_{\pi^\perp} = (AJ - JA)\big|_{\pi^\perp} = 0 \implies AJ\big|_{\pi^\perp} = JA\big|_{\pi^\perp}.$$

This completes the case where $\mathrm{Rank}(B) = 2$. By Lemma 3.2 above $\mathrm{Rank}(B)$ cannot be equal to 1. So if $\mathrm{Rank}(B) = 0$ then $B = 0$ and A and J commute everywhere, in particular they commute on the orthogonal complement of every complex line.

For the converse suppose that A and J commute on the orthogonal complement of a complex line π. Then it follows that $\pi^\perp \subset \ker(B)$ so we again have $\mathrm{Rank}(B) \leq 2$. If $\mathrm{Rank}(B) = 2$ then we have $\pi^\perp = \mathrm{Ker}(B)$ and since B is symmetric we can find an orthonormal basis of π, $\{e_1, Je_1\}$ so that

$$Be_1 = \lambda_1 e_1 \quad \text{and} \quad BJe_1 = -\lambda_1 Je_1 \qquad \lambda_1 \neq 0.$$

Extend the orthonormal basis of π to an orthonormal basis for the entire space V. Then up to the usual curvature symmetries the only non-zero entry of the curvature tensor R^S_B is

$$\begin{aligned} R^S_B(e_1, Je_1, Je_1, e_1) &= (Be_1, e_1)(BJe_1, BJe_1) - (Be_1, Je_1)(BJe_1, e_1) \\ &= -\lambda_1^2. \end{aligned}$$

Since $\mathrm{Rank}(B) = 2$ it follows that $\mathrm{Rank}(C) = 2$ also and since π is a complex line we have $C\pi = B\pi$. Hence up to the usual symmetries, computing the only non-zero entry of R^S_C we obtain

$$\begin{aligned} R^S_C(e_1, Je_1, Je_1, e_1) &= (Ce_1, e_1)(CJe_1, Je_1) - (Ce_1, Je_1)(CJe_1, e_1) \\ &= -(Ce_1, Je_1)^2 \\ &= -(JBe_1, Je_1)^2 \\ &= -(Be_1, e_1)^2 \\ &= -\lambda_1^2. \end{aligned}$$

Thus we conclude that $R^S_B = R^S_C$. Again by Lemma 3.2 we cannot have $\mathrm{Rank}(B) = 1$ so if $\mathrm{Rank}(B) = 0$ then $B = 0$ and therefore $C = JB = 0$ and $R^S_B = R^S_C = 0$ as desired. ☐

An interesting corollary of linear algebra is immediate from Theorem 3.1 and Lemma 3.2.

Corollary 3.1. *Let $A : V \to V$ be a linear map. Let $(V, (\cdot,\cdot))$ be an inner product space endowed an almost complex structure J. If $A^* = -A$ and if $AJ = -JA$ then* $\mathrm{Rank}(A) \neq 2$.

Proof. Assume for a contradiction that $\mathrm{Rank}(A) = 2$. Then by Lemma 3.1 the Gray Identity is satisfied for R^Λ_A. Hence Theorem 3.1 states we must have $AJ = JA$ and so $-JA = JA$. Since J is invertible we conclude A must be the zero map giving us a contradiction. ☐

We actually can say more. The corollary above can be viewed as a special case of the following lemma. We give a purely algebraic proof.

Lemma 3.4. *Let $(V, (\cdot,\cdot))$ be an inner product space. Let A, J be skew-adjoint linear maps on V with* $\det(J) \neq 0$. *If $AJ = -JA$ then* $\mathrm{Rank}(A) \neq 2$.

Proof. Let $\mathrm{Rank}(A) = 2$ and block diagonalize A so $\mathcal{B} := \{e_1, f_1, h_1, \ldots, h_t\}$ is an orthonormal basis for V with

$$\ker(A) = \mathrm{span}\{h_1, \ldots, h_t\} \qquad \text{and} \qquad Af_1 = \lambda e_1, \quad \lambda > 0.$$

Consider Jh_j for $j = 1, \ldots, t$,

$$Jh_j = (Jh_j, e_1)e_1 + (Jh_j, f_1)f_1 + \sum_{i=1}^{t}(Jh_j, h_i)h_i.$$

Hence,

$$AJh_j = -\lambda(Jh_j, e_1)f_1 + \lambda(Jh_j, f_1)e_1.$$

On the other hand since $h_j \in \ker(A)$ we have $JAh_j = 0$. Thus,

$$-\lambda(Jh_j, e_1)f_1 + \lambda(Jh_j, f_1)e_1 = 0.$$

Since e_1 and f_1 are linearly independent we conclude, $(Jh_j, e_1) = (Jh_j, f_1) = 0$ for $j = 1, \ldots, t$. Therefore since J is skew-adjoint we have that $\ker(A)$ and $\ker(A)^\perp$ are J invariant subspaces. The matrix of $J\big|_{\ker(A)^\perp}$ in the basis $\mathcal{B}$ is of the form

$$J\big|_{\ker(A)^\perp} = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \qquad \eta \neq 0.$$

Now see that $AJf_1 = \eta Ae_1 = -\eta\lambda f_1$ and that $JAf_1 = \lambda Je_1 = -\eta\lambda f_1$. But since $AJ = -JA$ we must have $\eta\lambda = 0$, but both η and λ were chosen to be non-zero, giving us our contradiction. □

At first glance it would seem as though Theorem 3.1 and Lemma 3.1 contradict each other in the case where A is a skew-adjoint rank 2 map that anti-commutes with the almost complex structure J. However as Lemma 3.4 demonstrates this is a vacuous statement since no such A exists.

Open Questions

1. What can be said about the general linear dependence of canonical algebraic curvature tensors built from skew-adjoint maps?

2. Can a canonical algebraic curvature tensor be geometrically realized on a curvature homogeneous Hermitian manifold?

3. What is the relationship between the sets

$$\text{span}\{R^S_A \,|\, A^* = A \ \text{ where } \ R^S_A \ \text{ satisfies the Gray Identity}\},$$

and

$$\text{span}\{R^\Lambda_A \,|\, A^* = -A \ \text{ where } \ R^\Lambda_A \ \text{ satisfies the Gray Identity}\}?$$

In particular do they span the set of all algebraic curvature tensors that are geometrically realizable by a Hermitian manifold?

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