The Three-Variable Bracket Polynomial for Reduced, Alternating Links

Kelsey Lafferty

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We first show that the three-variable bracket polynomial is an invariant for reduced, alternating links. We then try to find what the polynomial reveals about knots. We find that the polynomial gives the crossing number, a test for chirality, and in some cases, the twist number of a knot. The extreme degrees of d are also studied.

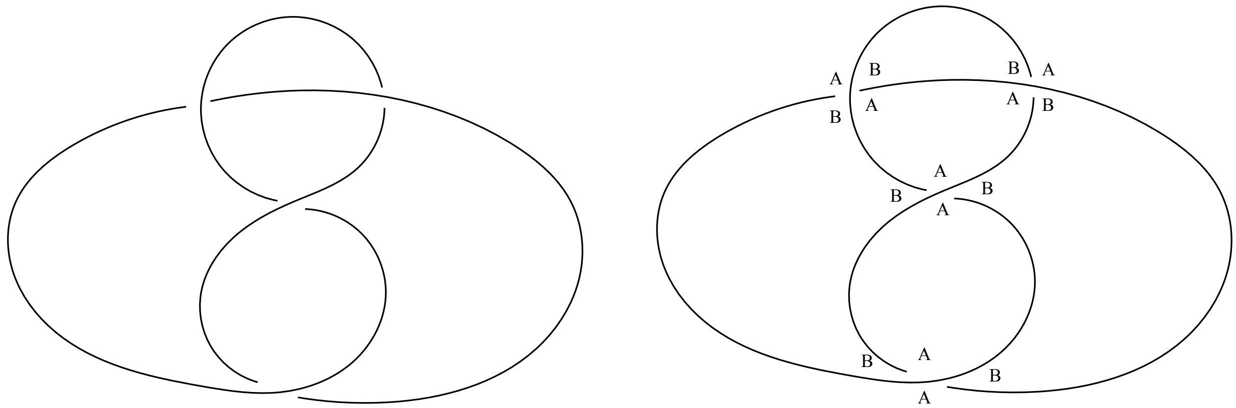
1 Introduction

Recently, the mathematical study of knots has gained popularity among mathematicians and scientists. Because invariants provide a test for whether or not two knots are isotopic, they play an important role in knot theory. In this paper, we show that the three-variable bracket polynomial is an invariant for reduced, alternating links. We then attempt to find what the three-variable bracket polynomial reveals about knots. For more about knots, links, and invariants, see [1].

For our purposes, a mathematical *knot* is any closed loop in $\mathbb{R}^3$. Knot diagrams are embeddings of three-dimensional knots into the plane, where over-crossings are represented by a solid line, and under-crossings are represented by a break in the line. A knot with no crossings is the *unknot*. A *link* is multiple knots that may or may not be joined in such a way that they can not be pulled apart without breaking one of them (see Figure 3(a)).

In this paper, the only knots considered are reduced, alternating knots. A knot is *reduced* if it cannot be redrawn with any fewer crossings. A knot is *alternating* if, were one to choose an arbitrary point on the knot and follow a line around the knot, every time one passed an over-crossing, the next crossing must be an under-crossing; similarly, under-crossings must be followed by over-crossings. An example of a reduced, alternating knot is shown in the knot diagram in Figure 1(a). A *twist* is a region of a knot in which only two strands cross each other. The *twist number* of an alternating knot diagram is the fewest number of twists in any diagram of the knot. In Figure 1(a), the uppermost left and uppermost right crossings could each be considered a twist. However, upon further examination, one can easily see that the two can be viewed as a single twist with two crossings.

One can define A and B regions of an alternating knot. *A regions* are defined as the region to the left of an under-crossing strand when viewing the crossing from that strand; the

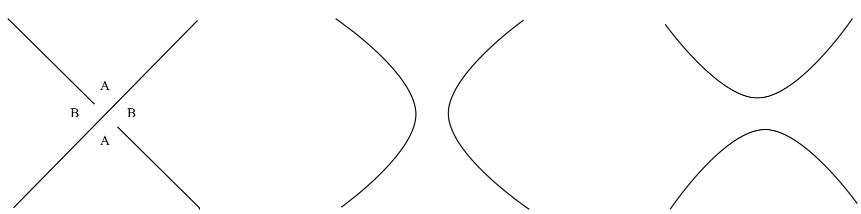


(a) A Reduced, Alternating, Figure-8 Knot

(b) Labeled A and B Regions for the Figure-8 Knot

Figure 1: Figure-8 Knot

B regions are to the right. An example of a knot with labeled regions is depicted in Figure 1(b). Using these regions, it is possible to color any reduced, alternating knot diagram with a checkerboard pattern by shading either the A or B regions. *Smoothings* are defined for each crossing as cutting the knot and glueing it so that either the A regions or B regions are connected at the crossing, as depicted in Figure 2. If the A regions are connected after the smoothing, then it is an A smoothing. Otherwise, it is a B smoothing. Another way to describe smoothings is by zero and infinity tangles. A *tangle* is any part of a knot where a loop can be drawn around it so that two strands are entering the loop and two are exiting. A zero tangle is a tangle where the strands enter from the right or left and exit the opposite side without any crossings. The "slope" of the strands is zero. Figure 2(c) shows a zero tangle. An infinity tangle is a tangle where two strands enter from the top or bottom and exit the opposite side without any crossings. The "slope" of the strands is infinite. An example of an infinity tangle is depicted in Figure 2(b). These concepts can be used to identify twists with multiple crossings. If a twist is aligned vertically (so that its crossings make a vertical line), then an A twist is a twist in which all A smoothings would give an infinity tangle; a B twist is a twist in which all B smoothings would give an infinity tangle. In Figure 12, the twist is a B twist.



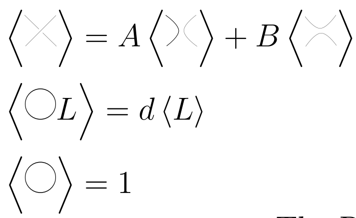
(a) Original Crossing

(b) A Smoothing of Crossing

(c) B Smoothing of Crossing

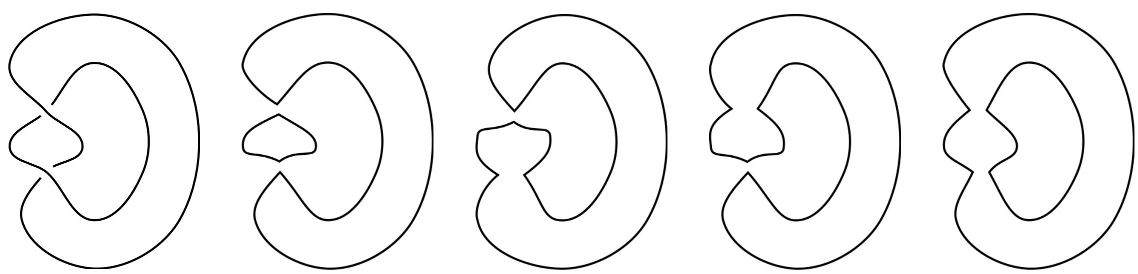
Figure 2: Smoothing Options for a Crossing

Using the notion of smoothings, polynomials can be derived from knots. The *three-variable bracket polynomial* turns out to be an invariant for certain links, meaning that if two links have different polynomials, then they are not isotopic. For the rest of this paper, we use the term bracket polynomial to refer to the three-variable bracket polynomial, although typically it refers to a one-variable polynomial. Often a recursive definition for the bracket polynomial is used. The recursive definition of the bracket polynomial for a link L is depicted below (for more on the bracket polynomial see [4]).



The Recursive Definition of the Bracket Polynomial

This paper uses the state model approach of defining the bracket polynomial. A *state* is a specific assigning of smoothings to crossings. The states of the Hopf Link are given in Figure 3. In the state model approach, one considers every possible state of the knot. Each state is given a term $A^m B^n d^p$, where m is the number of A smoothings, n is the number of B smoothings, p is one less than the number of loops created after the smoothings. By summing the terms from each state, one creates the bracket polynomial. This paper orders terms by decreasing degrees of A, with the largest degree of A coming first. An example of deriving the bracket polynomial is given in Figure 3 for the Hopf Link.



(a) Hopf Link (b) All A State: A^2d (c) One A, One B State: AB (d) One A, One B State: BA (e) All B State: B^2d

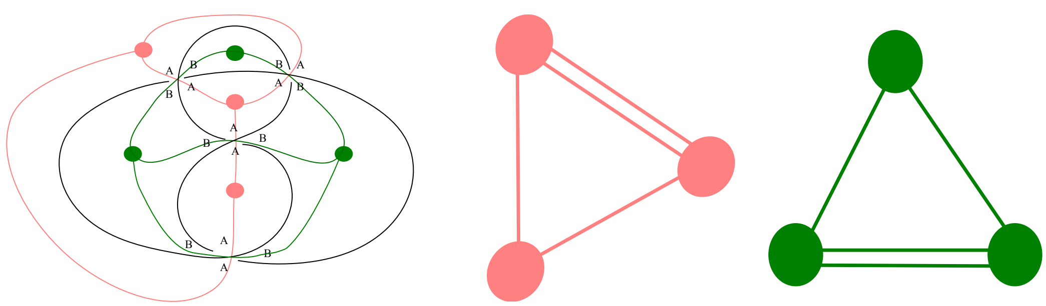
Figure 3: Bracket Polynomial $A^2d + 2AB + B^2d$

Note that if d were to equal one, the bracket polynomial would be equal to $(A+B)^r$ where r is the total number of crossings. The polynomial should follow the binomial expansion because one is assigning an A or B smoothing to each of the crossings. However, because d does not equal one, the bracket polynomial does not follow the binomial expansion exactly. For example, the bracket polynomial for the Figure-8 knot is given in expression (1).

$$A^4d^2 + 4A^3Bd + (5A^2B^2 + A^2B^2d^2) + 4AB^3d + B^4d^2 \tag{1}$$

We define a term that *splits* as one that has the same degrees of A and B, but different degrees of d. In (1), a term that splits is $(5A^2B^2 + A^2B^2d^2)$. Notice that if d equaled one, then the term would be $6A^2B^2$, which is what would be expected from the binomial expansion.

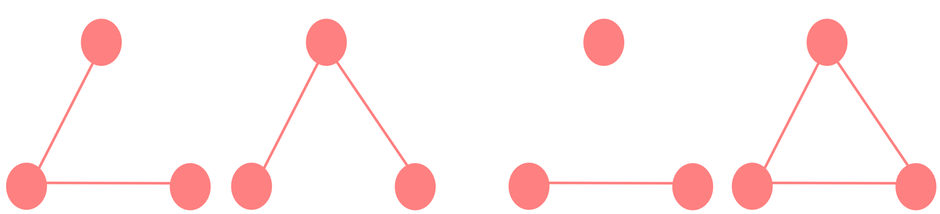
Closely connected to the states of a knot are the subgraphs of its connected graph. A connected graph can be obtained from a knot diagram by placing a vertex in each of the A regions, and connecting the vertices to other vertices with edges through crossings. One obtains the connected graph for the B regions by taking the planar dual of the graph for the A regions. An example is depicted in Figure 4. *Parallel edges* are edges that connect the same two vertices. Parallel edges can be reduced to *multi-edges*, single edges with numbers next to them indicating multiplicity. A *cycle* is any path from a vertex back to itself where the path does not include any of the same edges or vertices except for the first and last vertex. *Subgraphs* are any combination of vertices and edges from the connected graph. A *subtree* is a connected subgraph with no cycles. A *maximal subtree* is any subtree that touches every vertex; several examples are depicted in Figure 5. Notice that usually the two connected graphs obtained from the A and B regions are different. For more information on graphs, see [1] and [7].



(a) The Knot Diagram

(b) Connected Graphs

Figure 4: Creating Connected Graphs



(a) Maximal Subtrees

(b) Not Maximal Subtrees

Figure 5: Subtrees for Connected Graph in Figure 4

Kauffman notes that the subgraphs of a connected graph are in a one-to-one correspondence with the various states of a knot [4]. The state of the knot can be obtained by drawing a loops around the edges, including the vertices on either end. If two adjacent edges are included in the subgraph, draw one loop around both. If any vertex is not connected by any edges, draw a loop around that vertex. An example is depicted in Figure 6. Studying the connected graph of the knot can be thought of as studying the possibilities for states of the knots. The connected graph becomes a powerful tool to use in proofs of various properties of knots and their bracket polynomials.

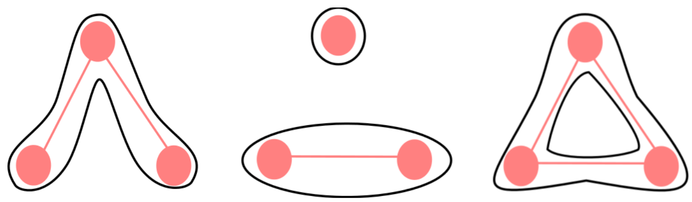
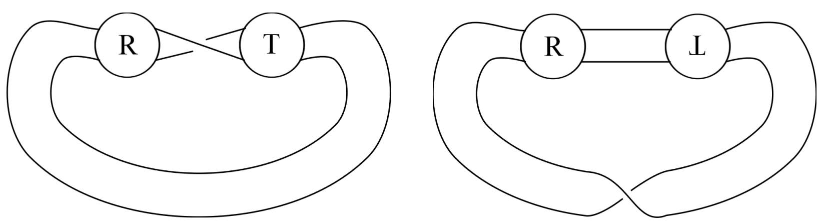


Figure 6: Obtaining States from Subgraphs

Rather than leaving the bracket polynomial as a three-variable expression, many change the bracket polynomial into a one-variable polynomial related to the Jones polynomial by using the conditions that $B = A^{-1}$ and $d = -A^2 - A^{-2}$. These conditions guarantee that the polynomial is invariant under Reidemeister II and III moves (see [1]). However, the bracket polynomial is a knot invariant by restricting the class of links under consideration. Before specifying this class, it is helpful to define flypes and mutations. A *flype* is when one flips a part of the knot, thereby moving crossings from one part of the knot to another. For an example of a flype, see Figure 7, in which R and T denote tangles.



(a) Before Flype (b) After Flype

Figure 7: Flype Example

Recall, a tangle is any part of a knot where a loop can be drawn around it so that two strands are entering the loop and two are exiting. A *mutation* is an operation in which a tangle is rotated by 180 degrees without affecting the rest of the knot. For further clarification, see Figure 8. In a mutation, the tangle is essentially cut from the knot and glued back into place. For more on flypes, tangles, and mutations, see [1].

Theorem 1. *The three-variable bracket polynomial is an invariant for reduced, alternating links.*

Proof. The Tait Flyping Theorem states that any two reduced, alternating diagrams are related by a sequence of flypes [5]. So, if the three-variable bracket polynomial is invariant

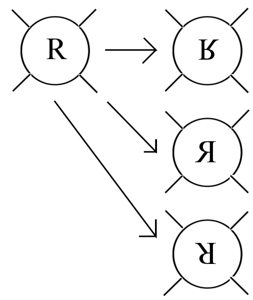
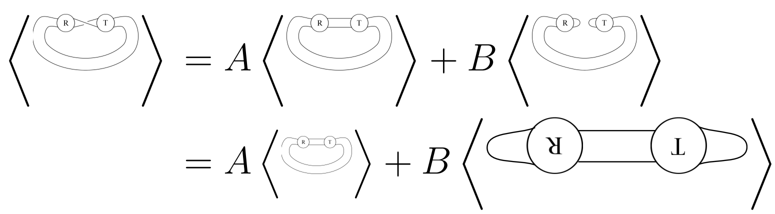
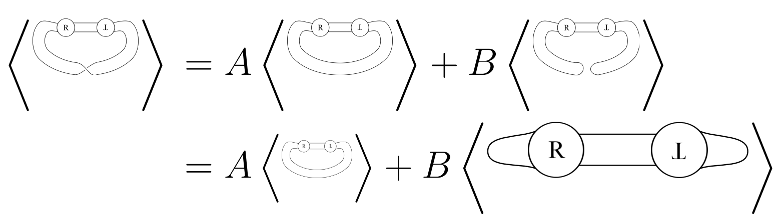


Figure 8: Mutations

under flypes, it must be an invariant of reduced, alternating knots. It helps to first prove that the three-variable polynomial is invariant under mutation. It is known that mutations preserve whether or not a knot is alternating, the A and B smoothings, and the number of components created in corresponding states. The number of components is preserved because even after mutation, zero and infinity tangles remain unchanged, thereby keeping constant the connectivity of the outer components. Hence, the three-variable bracket polynomial is invariant under mutation. Next, consider the three-variable bracket for a knot and the same knot after a flype.



(2)



(3)

Clearly, the diagrams within (2) and (3) are mutations of each other. Because the bracket polynomial is invariant under mutations, the two polynomials must be the same. Thus, the three-variable polynomial is invariant under flypes. □

2 The Bracket Polynomial

The goal of this research is to discover what the bracket polynomial can tell one about a reduced, alternating link. Specifically, it would be nice to extend some of the known facts about the one-variable polynomial or Jones polynomial to the bracket polynomial. For example, the span of the Jones polynomial is equal to the crossing number of the knot [7]. Similarly, the sum of the absolute values of the penultimate coefficients of the Jones polynomial gives the twist number of the knot [2]. To this end, the extreme degrees of d are studied after a few simple results.

2.1 Crossing Number

To begin, the crossing number of the knot is the sum of the degrees of A and B for any given term of the polynomial. The knot is already reduced and alternating, and the degrees of A and B in any given term indicate the number of A smoothings and B smoothings applied. Because each crossing is smoothed in some way, the sum of the degrees of A and B must give the total crossing number.

Lemma 1. *The crossing number of a knot is equal to the sum of the degrees of d from the A^0 and B^0 terms.*

Proof. The total number of regions of a knot is two greater than the crossing number. This follows from viewing the knot diagram as a four-valent planar graph and applying the Euler characteristic. (Note that in this case, vertices for the planar graph replace crossings, and edges replace strands between crossings. This is not the connected graph from Section 1.) The A^0 and B^0 terms isolate the A and B regions, respectively. However, the factors of d count one less than the number of components created. Essentially, the factors of d count one less than the A and B regions in each term. Thus, the sum of the degrees of d for those two terms gives two less than the total number of regions, which is just the crossing number. □

2.2 Test for Chirality

The Bracket polynomial also provides a test for the chirality of a knot. A knot is *chiral* if it is not isotopic to its mirror image. A knot is *achiral* if it is isotopic to its mirror image.

Theorem 2. *If interchanging A and B in the three-variable bracket polynomial of a knot changes the polynomial, then the knot is chiral. Equivalently, if a knot is achiral, then its bracket polynomial must be invariant under interchanging A and B.*

Proof. Consider a crossing compared to its mirror image. As depicted in Figure 9, the A and B regions switch. Because the state-model approach to the bracket polynomial gives the same number of components regardless of whether a specific strand goes over or under at a crossing, the three-variable bracket polynomial for the mirror image k_2 of a knot k_1 is equivalent to the bracket polynomial for k_1 where the A's and B's are interchanged. If k_1 is achiral, then it is isotopic to k_2. This means that the three-variable bracket polynomials are the same, even though the A's were replaced by B's and the B's by A's. Thus, if a knot is achiral, its bracket polynomial is invariant under switching A and B. □

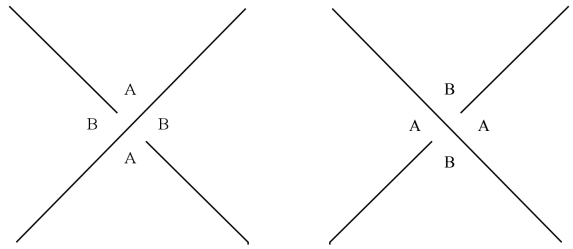


Figure 9: A Crossing and its Mirror Image

Van Quach Hongler shows that if an alternating link is achiral, then it must have an equal number of shaded and unshaded regions in its checkerboard diagram [8]. The same result follows from Theorem 2. If one can interchange A and B, then the degrees of d must be equal for the A^0 and B^0 terms in the bracket polynomial. The degrees of d for those terms represent one less than the total number of A and B regions, respectively. Because the A and B regions correspond to shaded and unshaded regions, the number of shaded and unshaded regions must be equal if an alternating link is achiral.

2.3 The Term with the Minimum Degree of d

One of the interesting terms in the polynomial is the d^0 term, which corresponds to the types of smoothings that result in the unknot.

2.3.1 Uniqueness of Term and Meaning of Coefficient

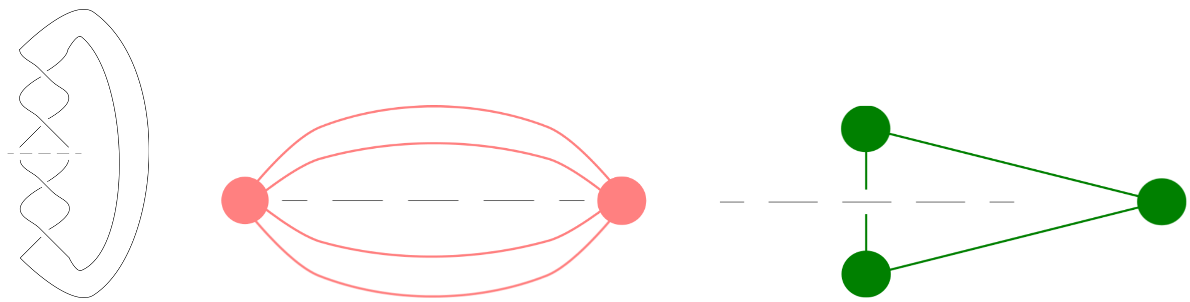
Lemma 2. *The d^0 term of the bracket polynomial is unique.*

Proof. Uniqueness can be proved with the use of a connected graph. Kauffman notes that Jordan-Euler trails, which correspond to unknots created by smoothing every crossing, are in a one-to-one correspondence with maximal subtrees of the connected graph [4] (Kauffman states that a proof is in [3]). It is known that any tree with n vertices has $n-1$ edges, so maximal subtrees must have the same number of edges. Now, the edges that are part of the maximal tree correspond to degrees of either A or B depending on the diagram, and the edges not part of the maximal tree correspond to the other. Because maximal trees have the same number of edges, the degrees of A and B must be constant for any option of smoothings for the crossings of a knot that produces the unknot. Thus, there can only be one d^0 term in the bracket polynomial. □

From this, it becomes apparent that the coefficient of the d^0 term is the number of maximal subtrees in the connected graph. We reason as follows: the coefficient is the number of ways that the specific combination of A and B smoothings creates the unknot. These combinations correspond to Jordan-Euler trails, which are in a one-to-one correspondence with the maximal subtrees of the connected graph of the knot [4]. So, the coefficient of the d^0 term simply gives the number of maximal subtrees of the connected graph.

2.3.2 (2, c) Torus Links

The uniqueness of the minimum degree of d plays a role in the proof of a test for (2, c) torus links. *Torus links* are links that can be embedded on the surface of a torus without intersection. (2, c) torus links describe a specific type of torus knots. In terms of the knot diagrams, (2, c) torus links are knots that have only one twist. For more on torus links, see [1]. Figure 10 shows a link with one twist along with its connected graphs. The bracket polynomial provides a simple test for whether or not a link has one twist. First, recall that splitting terms occur where the same numbers of A and B smoothings give different degrees of d.



(a) A (2, c) Torus Link

(b) Connected Graph for A Regions

(c) ConnectedGraph for B Regions

Figure 10: A (2, c) Torus Link and its Connected Graphs. Any number of crossings, vertices, and edges can occur in the region represented by the dashed line.

Theorem 3. *A knot has one twist if and only if no terms split in its three-variable bracket polynomial.*

Proof. Let k_1 be a knot with c crossings and one twist. Consider the $A^m B^n d^p$ term. First, if m or n equals zero, the terms do not split. Suppose, without loss of generality, that the single twist is a B twist. Consider the situation in which one switches an A and a B smoothing in any state of k_1 with m A smoothings and n B smoothings. Changing the B smoothing to an A smoothing would separate two regions, increasing the total number of loops in the state by one, which means the degree of d is now $p+1$. However, simultaneously, the A smoothing changes to a B smoothing. The B smoothing replacing the A smoothing connects two regions that were previously separated, decreasing the number of loops and the degree of d by one. So, the degree of d is now $p+1-1$, which is just p. Hence, regardless of the placement of smoothings within a knot of one twist, the degree of d for any given term does not vary.

Next, suppose that none of the terms split in the bracket polynomial of a knot k_2. Assume that k_2 has more than one twist. The connected graphs for knots with one twist are a single cycle and a single multi-edge. Because k_2 has more than one twist, its connected graphs cannot be either of these. Consider a maximal subtree of the connected graph for k_2. Let q be the number of edges in this maximal subtree. Because no terms split, one can choose any

q edges in the connected graph to create a maximal subtree (see Lemma 2 and its proof). However, a single cycle and a single multi-edge are the only connected graphs that satisfy this condition. Therefore, k_2 cannot have more than one twist, which is a contradiction. Thus, a knot has one twist if and only if no terms split in its three-variable bracket polynomial. □

Corollary 1. *If a knot has more than one twist, the term that contains the d^0 term always splits.*

Proof. Suppose a knot has more than one twist. Again, its connected graphs cannnot be a single cycle or a single multi-edge. If one assumes that the d^0 term does not split, the rest of the proof is identical to the proof by contradiction for Theorem 3. □

2.4 The Term with the Maximum Degree of d

Like the d^0 term, the term with the maximum degree of d is also special in the bracket polynomial.

2.4.1 Uniqueness and Position of the Term

To start, the term with the maximum degree of d is not unique in the bracket polynomial. Consider the bracket polynomial for the figure-8 knot (1), the maximum degree of d is 2, which occurs three different times in the polynomial. Although the term is not unique, the position of the terms in the polynomial is subject to restrictions.

Theorem 4. *The maximum degree of d must occur in the term where the degree of A is zero, the degree of B is zero, or in a term that splits.*

Proof. To begin, it is known that the maximum degree of d can occur where either the degree of A is zero, such as in the case of the right-handed trefoil (3), or the degree of B is zero, such as in the case of the left-handed trefoil (4).

$$A^3d + 3A^2B + 3AB^2d + B^3d^2 \tag{3}$$

$$A^3d^2 + 3A^2Bd + 3AB^2 + B^3d \tag{4}$$

Suppose the maximum degree of d does not occur in a term where the degree of either A or B is zero. In the corresponding state, there must be multiple isolated A and B regions. An isolated region is the area inside or outside a loop from the knot. The case where there are only two isolated regions represents the unknot. We will assume there are at least three isolated regions and that at least two of these regions are A regions. Two cases must be considered, where there is only one isolated B region, and where there are multiple isolated B regions.

First, consider the case where there are multiple isolated B regions. In this case, there must be at least one B smoothing between the two isolated A regions and at least one A smoothing between the two isolated B regions. These two smoothings could be switched, which would connect the two A regions and connect the two B regions. Thus, the degree of d would decrease by two, but the degrees of A and B would remain the same. By definition, the term describing this state of the knot must split. So, the maximum degree of d occurs in a term that splits.

Next, consider the case where there is only one isolated B region, and the knot is not in the all A or all B state. This case is depicted below in Figure 11, note that where three isolated A regions are depicted within the isolated B region, there could be any number greater than zero of A regions. In this case, there must be at least one A smoothing, connecting two A regions, because it is assumed that the knot is not in the all B state. If this A smoothing were to change to a B smoothing, the two regions would be separated, increasing the power of d by one. This contradicts our assumption that the power of d is at its maximum in this term. Therefore, the maximum power of d must occur in a term where the degree of A is zero, the degree of B is zero, or in a term that splits. □

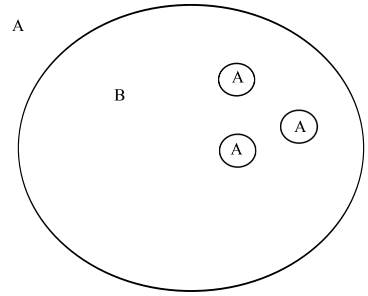


Figure 11: One Isolated B Region

2.4.2 Twist Number for Two-Bridge Knots

The term with the maximum degree of d can be used to find the twist number for two-bridge knots (defined below). However a couple of definitions and constructions must come first. To begin, define *smoothing across the twists* as the state obtained by choosing smoothings as in Figure 12. For example, if there is an A twist, then a B smoothing would be assigned to each of the crossings included in the twist.

Theorem 5. *For a knot with at least two crossings per twist, smoothing across the twists gives the maximum degree of d.*

Proof. Consider a twist with two crossings. The possibilities of smoothings are shown in Figure 13.

Clearly, options B and C are not the best choices to maximize the number of loops created, which also maximizes the degree of d, because option A has one more loop than options B and C. So, the only options to consider for maximizing the number of loops created are A and D. Suppose that a knot is in a state with the maximal number of loops. As has already been noted, each twist must have been replaced by either option A or D. Replacing

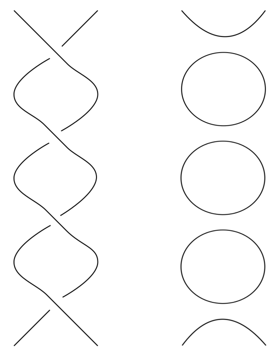
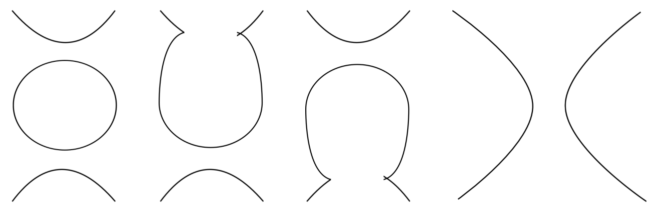


Figure 12: Smoothing Across the Twists



(a) A (b) B (c) C (d) D

Figure 13: Options for a Twist with Two Crossings

an infinity tangle with a zero tangle at worst will connect two loops, decreasing the total number of loops by one (of course, it could also separate two loops and increase the number). However, option A also has at least one smaller loop in the middle of the zero tangle, which increases the total number of loops by at least one. So replacing any option D with option A will keep the same or increase the number of total loops. Thus, by applying this reasoning to every twist that had been replaced by option D, one finds that replacing every twist with option A gives the maximal number of loops. □

A *two-bridge knot* is a knot for which a plane can be found to intersect the knot in four points, and the parts of the knot on either side of the plane can be isotoped into the plane without any intersections. Two-bridge knots with odd numbers of twists can be drawn as shown below in Figure 14, where each box represents a twist. If a two-bridge knot has an even number of twists, the twists are connected in a slightly different manner.

Theorem 6. *Two-bridge knots can be smoothed into the unknot when every twist is replaced by a zero tangle.*

Proof. The proof for this theorem will be split into two cases, when the knot has an even number of twists, and when the knot has an odd number of twists. First, consider the base cases, two-bridge knots with one, two, or three twists. As shown in Figure 15, in each case, having all zero tangles gives the unknot.

Next, consider the inductive cases for knots with even and odd numbers of twists. Let n be the number of twists in the knot where all zero tangles gives the unknot. Suppose

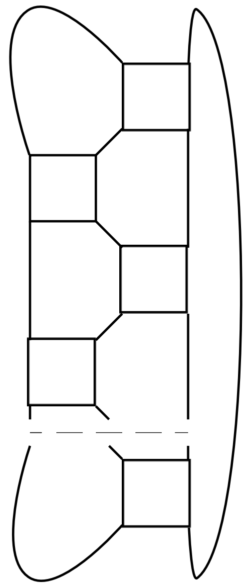
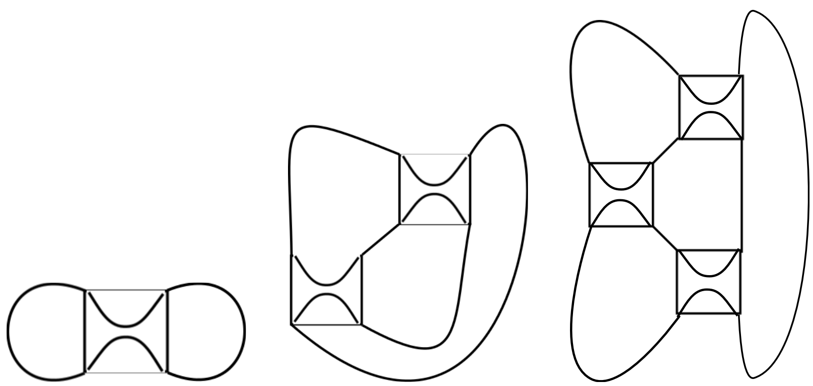
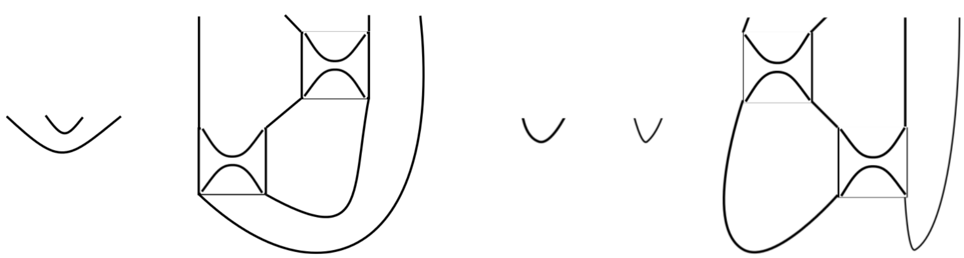


Figure 14: A 2-Bridge Knot with an Odd Number of Twists. Any even number of boxes can occur in the region represented by the dashed line.



(a) One Twist (b) Two Twists (c) Three Twists

Figure 15: Base Cases



(a) Even Number of Twists (b) Odd Number of Twists

Figure 16: Inductive Cases

two new twists are added to the bottom of the knot, and suppose that the two twists are smoothed in such a way that they are zero tangles. Figure 16 shows that in both cases the bottom of the knot with $n+2$ twists is isotopic to the bottom of the knot with n twists. Thus, for all two-bridge knots, all zero tangles creates the unknot. □

Theorem 7. *For a two-bridge knot with n twists, c total crossings, and at least two crossings in each twist, if m is the maximum degree of d in the bracket polynomial, then $c - m = n$.*

Proof. Suppose we have a two-bridge knot with n twists, c total crossings, at least two crossings in each twist, and suppose m is the maximum degree of d in the bracket polynomial. Let c_i be the number of crossings in the ith twist. Notice that in a twist with c_i crossings, smoothing across the twist gives a zero tangle with $c_i - 1$ loops in between the two strands composing the zero tangle. From Theorem 5, smoothing across the twists gives the maximum degree of d. So, smoothing across the twists of our two-bridge knot, where l is the number of loops created, we have

$$l = \left(\sum_{i=1}^{n} c_i - 1 \right) + 1 \tag{5}$$

The additional 1 comes from the loop created by the zero tangles (see Theorem 6). By definition, $l = m + 1$. Substituting into (5) and simplifying,

$$m = \left(\sum_{i=1}^{n} c_i \right) - n \tag{6}$$

By definition, $c = \sum_{i=1}^{n} c_i$. Substituting into (6), we have $m = c - n$. Hence, $c - m = c - (c - n) = n$. □

Some of these results for two-bridge knots were discovered independently by Overduin, using different methods [6].

3 Conclusions

The three-variable bracket polynomial has potential to reveal much about reduced, alternating links. Further research may be conducted in several areas. First, we are fairly certain that a knot with only one crossing in each twist must have the maximum degree of d in the all A or all B state, but this needs proof. Further, it may be true that the maximum degree of d occurring in the all A state, all B state, or both, but nowhere else, implies that the link either has one twist or one crossing in each twist. Beyond these, it would be interesting to find a stronger test for chirality, the pattern of the splitting terms, what the other degrees of d reveal about the knot, and if the bracket polynomial gives the twist number for links in general.

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