# Curvature homogeneity of type $(1,3)$ in pseudo-Riemannian manifolds

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**Abstract**

We construct two new families of pseudo-Riemannian manifolds which are curvature homegeneous of type $(1,3)$. The first family given has signature $(2k, 2k+1)$ and is curvature homogeneous of type $(1,3)$ but not curvature homogeneous. The second family given has signature $(1,2)$ and is curvature homogeneous of type $(1,3)$ of all orders but not locally homogeneous, showing there is no finite Singer number for this type of curvature homogeneity.

## 1 Introduction

Let $(M, g)$ be a pseudo-Riemannian manifold, and $\nabla$ be its Levi-Civita connection. We begin with some preliminaries and definitions.

### 1.1 Curvature Homogeneous Manifolds

Let $\mathcal{X}$ denote the ring of smooth vector fields on $M$. Let $TM$ denote the tangent bundle of $M$, and $T^*M$ its dual. The Riemann curvature tensor $R$ of type (1,3), that is, $R \in TM \otimes (T^*M)^3$ is then given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

where $X, Y, Z \in \mathcal{X}$. The Riemann curvature tensor $\mathcal{R}$ is a tensor of type (0, 4), that is, $\mathcal{R} \in \otimes^4(T^*M)$, and is given by

$$\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z, W). \tag{2}$$

Let $p \in M$, and $T_pM$ denote the tangent space of $M$ at $p$. Singer [8] defines a pseudo-Riemannian manifold to be *curvature homogeneous* (CH) if for any points $p, q \in M$, there exists a linear isometry $F : T_pM \to T_qM$ such that the pullback $F^*$ satisfies $F^*g_q = g_p$ and $F^*\mathcal{R}_q = \mathcal{R}_p$. Let $\nabla^r\mathcal{R}$ denote the $r$-th covariant derivative of $\mathcal{R}$. A pseudo-Riemannian manifold is *curvature homogeneous up to order* $n$ ($\mathrm{CH}_n$) if there exists a linear isometry $F$ which also satisfies $F^*\nabla^r\mathcal{R}_q = \nabla^r\mathcal{R}_p$ for all $r \leq n$.

Singer proved the following theorem in the Riemannian setting, which motivated the study of curvature homogeneity [8].

**Theorem 1.1.** *Let $\mathcal{M}$ be an $m$-dimensional manifold. There exists an integer $k_m$ so that if $\mathcal{M}$ is curvature homogeneous of order $k_m$, $\mathcal{M}$ is locally homogeneous.*

Later, Podesta and Spiro proved an analogous theorem for the pseudo-Riemannian case [7].

**Theorem 1.2.** *Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$. There exists an integer $k_{p,q}$ so that if $(M, g)$ is curvature homogeneous of order $k_{p,q}$, $(M, g)$ is locally homogeneous.*

A generalization of this is given by Kowalski and Vanžurová in [5] and [6]. A pseudo-Riemannian manifold is *curvature homogeneous of type* (1,3) (CH(1,3)) if for any pair $p, q \in M$, there exists a linear homothety $\phi : T_pM \to T_qM$ such that the pullback $\phi^*$ satisfies $\phi^* R_q = R_p$. We say that the manifold is *curvature homogeneous of type (1,3) of order* $n$ ($\text{CH}_n(1,3)$) if for every $r \leq n$ there exists a linear homothety $\phi_r$ whose pullback satisfies $\phi_r^* \nabla^r R_q = \nabla^r R_p$.

There has been quite a bit of work done on curvature homogeneous manifolds of higher signatures. The cases most closely related to this paper include the ones which are of balanced signature researched by Dunn, Gilkey, and Nikčević in [3] and [2], as well as those which are nearly balanced researched by Dunn in [1]. CH(1,3) manifolds have been researched only in the Riemannian case by Kowalski and Vanžurová in [5] and [6]. We will make use of the following results in [6]:

**Theorem 1.3.** *Let $(M, g)$ be a pseudo-Riemannian manifold of dimension $m$, and suppose that for every $0 \leq r \leq n$ there exists and with an orthonormal moving frame $\{E_1, \dots E_m\}$ on $M$, such that for fixed $p \in M$ and any $q \in M$, $\nabla^r \mathcal{R}(E_i, E_j, E_k, E_l; E_{s_1}, \dots, E_{s_r})|_q = \Theta_r(q) \nabla^r \mathcal{R}(E_i, E_j, E_k, E_l; E_{s_1}, \dots, E_{s_r})|_p$ where $\Theta_r$ is a smooth positive function. Then $(M, g)$ is $CH_n(1,3)$.*

## 1.2 Model Spaces

Let $V$ be a finite dimensional real vector space. A bilinear symmetric function $\sigma : V \to \mathbb{R}$ is an inner product on $V$ if for every non-zero vector $v$, there exists some vector $w$ such that $\sigma(v, w) \neq 0$.

We say a multilinear function $R : V^4 \to \mathbb{R}$ is an *algebraic curvature tensor* if

$$R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y),$$
$$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0.$$

We call the collection $(V, \sigma, R)$ a *model space*. The collection $(V, R)$ is called a *weak model space*. For example, for any $p \in M$, $(T_pM, g|_p, R|_p)$ is a model space, where $g|_p$ and $R|_p$ denotes the metric and curvature at $p$, respectively, and $(T_pM, R|_p)$ is a weak model space.

Consider two model spaces $(V, \sigma, R)$ and $(W, \upsilon, S)$. These model spaces are *isomorphic* if there exists an isomorphism of vector spaces $H : V \to W$ which satisfies

$$\begin{aligned} \sigma(x, y) &= \upsilon(H(x), H(y)), \\ R(x, y, z, w) &= S(H(x), H(y), H(z), H(w)). \end{aligned}$$

Two weak model spaces $(V, R)$ $(W, S)$ are isomorphic if there exists an isomorphism of vector spaces $G : V \to W$ which satisfies

$$R(x, y, z, w) = S(G(x), G(y), G(z), G(w)).$$

Let $\varphi$ be a symmetric bilinear form on $V$. $R_\varphi$ is an algebraic curvature tensor given by

$$R_\varphi(x, y, z, w) = \varphi(x, w)\varphi(y, z) - \varphi(x, z)\varphi(y, w).$$

If there exists such a symmetric bilinear $\varrho$ so that $R = \pm R_\varrho$, we say $R$ is a *canonical algebraic curvature tensor*. The following theorem may be found in [4].

**Theorem 1.4.** *Let $\varphi$ be a symmetric bilinear form with rank greater than or equal to 3. If $R_\varphi = R_\varrho$, $\varphi = \pm\varrho$.*

It is evident that a space is curvature homogeneous if and only if the model spaces at every pair of points are isomorphic. This motivates another kind of curvature homogeneity. A pseudo-Riemannian manifold is *weakly curvature homogeneous* if the weak model space at every pair of points are isomorphic.

## 1.3 Weyl Scalar Invariants

Let $\{x_1, \ldots, x_m\}$ be local coordinates on a pseudo-Riemannian manifold $(M, g)$. Adopt the Einstein convention of summing over repeated indices, and denote

$$\nabla_{\partial_{j_1}} \ldots \nabla_{\partial_{j_r}} R(\partial x_{i_1}, \partial_{x_{i_2}})\partial_{x_{i_3}} = R^{i_4}_{i_1 i_2 i_3 j_1 \ldots j_r} \partial_{x_{i_4}}.$$

We may construct scalar invariants by contracting all indices with other curvature entries or with the metric tensors $g_{ij}$ and $g^{ij}$. A scalar invariant formed in this method is a *Weyl scalar invariant*. They were defined by Weyl in [9].

For our purposes, we will be most interested in the scalar curvature $\tau$, the norm of the Ricci tensor $|\rho|^2$, and the norm of the curvature tensor $|R|^2$. These scalars are defined by

$$\begin{aligned} \tau &= g^{ij} R^k_{kij}, \\ |\rho|^2 &= g^{i_1 j_1} g^{i_2 j_2} R^k_{k i_1 j_1} R^l_{l i_2 j_2}, \\ |R|^2 &= g^{i_1 j_1} g^{i_2 j_2} g^{i_3 j_3} g_{i_4 j_4} R^{i_4}_{i_1 i_2 i_3} R^{j_4}_{j_1 j_2 j_3}. \end{aligned}$$

### 1.4 Geodesics

A curve $\gamma : [a,b] \subset \mathbb{R} \to M$ is a *geodesic at* $t_0$ if $\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt}|_{t=t_0} = 0$. We say $\gamma$ is a geodesic if it is a geodesic at all $t \in [a,b]$. If every geodesic in a pseudo-Riemannian manifold extends for infinite time, we say the manifold is *complete*.

### 1.5 The Manifold $M_{f,h}$

Let $M_{f,h} = (\mathbb{R}^{4k+1}, g_{f,h})$, where $\mathbb{R}^{4k+1}$ has coordinates given by $(x_0, x_1, \ldots, x_{2k}, y_1, \ldots y_{2k})$, and the metric $g_{f,h}$ has non-zero entries:

$$\begin{aligned} g_{f,h}(\partial_{x_0}, \partial_{x_0}) &= e^{2f(x_1)}, & & \\ g_{f,h}(\partial_{x_{2i-1}}, \partial_{x_{2i}}) &= 2h(x_0), & \quad & 1 \le i \le k, \\ g_{f,h}(\partial_{x_s}, \partial_{y_s}) &= 1, & & 1 \le s \le 2k, \end{aligned}$$

where $f$ and $h$ are smooth functions. We prove the following theorem in Section 2.

**Theorem 1.5.** *Let $h'$ and $h''$ be non-vanishing.*

*(a) $\tau$, $|\rho|^2$, and $|R|^2$ all vanish on $M_{f,h}$.*

*(b) $M_{f,h}$ is CH(1,3) if and only if $f'' = -2f'$.*

*(c) $M_{f,h}$ is CH if and only if $h'' \neq h'$.*

*(d) If $M_{f,h}$ is CH(1,3), then it is not complete if $f'$ is non-vanishing.*

### 1.6 The Manifold $M_f$

Let $M_f = (\mathbb{R}^3, g_f)$, where $\mathbb{R}^3$ has coordinates given by $(x_0, x_1, y_1)$, and the metric $g_f$ has non-zero entries:

$$\begin{aligned} g_f(\partial_{x_0}, \partial_{x_0}) &= e^{2f(x_1)}, \\ g_f(\partial_{x_1}, \partial_{y_1}) &= 1, \end{aligned}$$

where $f$ is smooth. We prove the following theorem in Section 3.

**Theorem 1.6.** *Suppose $(f')^2 \neq -f''$.*

*(a) $\tau$, $|\rho|^2$, and $|R|^2$ all vanish on $M_f$.*

*(b) $M_f$ is CH(1,3) of all orders.*

*(c) $M_f$ is weakly $CH_2$ if and only if $(f')^2 + f''$ is constant.*

## 2 The Geometry of $M_{f,h}$

**Lemma 2.1.** *For $M_{f,h}$,*

*(a) The non-zero Christoffel symbols are given by:*

$$\begin{aligned}
\nabla_{\partial_{x_0}}\partial_{x_0} &= -f'e^{2f}\partial_{y_1}, \\
\nabla_{\partial_{x_0}}\partial_{x_1} &= f'\partial_{x_0} + h'\partial_{y_2}, && 1 \le i \le k \\
\nabla_{\partial_{x_0}}\partial_{x_{2i}} &= h'\partial_{y_{2i-1}}, \\
\nabla_{\partial_{x_0}}\partial_{x_{2l-1}} &= h'\partial_{y_{2l}}, && 2 \le l \le k, \\
\nabla_{\partial_{x_{2i-1}}}\partial_{x_{2i}} &= -\frac{h'}{e^{2f}}\partial_{x_0}.
\end{aligned}$$

*(b) Up to the usual $\mathbb{Z}_2$ symmetries, the non-zero entries of $\mathcal{R}$ are given by:*

$$\begin{aligned}
\mathcal{R}_{1001} &= -e^{2f}((f')^2 + f''), \\
\mathcal{R}_{(2i-1)00(2i)} &= -h'', \\
\mathcal{R}_{1(2i)0(2i-1)} &= -f'h', \\
\mathcal{R}_{1(2l-1)0(2l)} &= -f'h', \\
\mathcal{R}_{(2i-1)(2j)(2i)(2j-1)} &= \frac{(h')^2}{e^{2f}}, && 1 \le j \le k, \\
\mathcal{R}_{(2i-1)(2j-1)(2i)(2j)} &= (1-\delta_{ij})\frac{(h')^2}{e^{2f}},
\end{aligned}$$

*where all indices denote subscripts of a $\partial_{x_*}$.*

*Proof.* The proof is evident, using the Christoffel symbols of the second kind, characterized by

$$\Gamma_{ijk} = g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}),$$

and may be calculated by

$$2\Gamma_{ijk} = \partial_{x_i} g(\partial_{x_j}, \partial_{x_k}) + \partial_{x_j}(\partial_{x_i}, \partial_{x_k}) - \partial_{x_k} g(\partial_{x_i}, \partial_{x_j}). \tag{3}$$

From here, we may simply use equations (1) and (2) to calculate the curvature. □

There are two moving frames $\beta_1, \beta_2$ which will be particularly useful. Let $\beta_1 = \{X_0, X_1, \ldots, Y_1, \ldots, Y_{2k}\}$, where

$$\begin{aligned}
X_0 &= \frac{\partial_{x_0}}{e^f}, \\
X_{2i-1} &= \lambda_{2i-1}(\partial_{x_{2i-1}} - h\partial_{y_{2i}}), \\
X_{2i} &= \lambda_{2i}(\partial_{x_{2i}} - h\partial_{y_{2i-1}}), \\
Y_s &= \frac{1}{\lambda_s}\partial_{y_s},
\end{aligned}$$

where $\lambda_s$ is a smooth, non-vanishing function on $M_{f,h}$. Let $\beta_2 = \{\bar{X}_0, \bar{X}_1, \dots, \bar{Y}_1, \dots, \bar{Y}_{2k}\}$, where

$$\begin{aligned} \bar{X}_0 &= X_0, \\ \bar{X}_s &= \frac{1}{\sqrt{2}}(X_s + Y_s), \\ \bar{Y}_s &= \frac{1}{\sqrt{2}}(X_s - Y_s). \end{aligned}$$

Note that $\beta_2$ is an orthonormal frame. In the frame $\beta_2$, the curvature entries are given by:

$$\mathcal{R}_{1001} = -\frac{\lambda_1^2}{2}((f')^2 + f''), \tag{4}$$

$$\mathcal{R}_{(2i-1)00(2i)} = -\frac{\lambda_{2i-1}\lambda_{2i}}{2}\frac{h''}{e^{2f}}, \tag{5}$$

$$\mathcal{R}_{1(2i)0(2i-1)} = -\frac{\lambda_1\lambda_{2i-1}\lambda_{2i}}{2\sqrt{2}}\frac{f'h'}{e^f}, \tag{6}$$

$$\mathcal{R}_{1_1(2l-1)0(2l)} = -\frac{\lambda_1\lambda_{2l-1}\lambda_{2s}}{2\sqrt{2}}\frac{f'h'}{e^f}, \qquad l \neq 1, \tag{7}$$

$$\mathcal{R}_{(2i-1)(2j)(2i)(2j-1)} = \frac{\lambda_{2i-1}\lambda_{2i}\lambda_{2j-1}\lambda_{2j}}{4}\frac{(h')^2}{e^{2f}}, \tag{8}$$

$$\mathcal{R}_{(2i-1)(2j-1)(2i)(2j)} = \frac{\lambda_{2i-1}\lambda_{2i}\lambda_{2j-1}\lambda_{2j}}{4}(1-\delta_{ij})\frac{(h')^2}{e^{2f}}, \tag{9}$$

where each subscript may be the subscript of either a $\bar{X}$ or $\bar{Y}$.

## 2.1 A few Weyl Scalar Invariants on $M_{f,h}$

*Proof of Theorem 1.5 (a).* We note that it is simpler to calculate the listed invariants in the orthonormal frame, as then we need only consider terms which have $g_{ss}$ or $g^{ss}$. Then one must take care to calculate the entries of the (1, 3) tensor. This may be done quite readily from the entries of the (0, 4) tensor, as the metric is reduced to an orthonormal one. It is then immediate that $\tau$, $|\rho|^2$, and $|R|^2$ vanish, as any curvature entry which uses an $\bar{X}$ will be cancelled out by the entry with a corresponding $\bar{Y}$ in that spot, as $g(X_s, X_s) = 1$ and $g(Y_s, Y_s) = -1$. To demonstrate this, we will look specifically at $\tau$.

Using our orthonormal frame, $\tau$ is given by:

$$
\begin{aligned}
\tau &= \sum_{s,r} g^{ss} g^{rr} \mathcal{R}_{rssr} \\
&= g(\bar{X}_0, \bar{X}_0)\mathcal{R}_{1001}(g(\bar{X}_1, \bar{X}_1) + g(\bar{Y}_1, \bar{Y}_1)) \\
&+ g(\bar{X}_0, \bar{X}_0)\mathcal{R}_{0110}(g(\bar{X}_1, \bar{X}_1) + g(\bar{Y}_1, \bar{Y}_1)) \\
&+ \sum_{i \leq k} \mathcal{R}_{(2i)(2i-1)(2i-1)(2i)} g(\bar{X}_{2i-1}, \bar{X}_{2i-1})(g(\bar{X}_{2i}, \bar{X}_{2i}) + g(\bar{Y}_{2i}, \bar{Y}_{2i})) \\
&+ \sum_{i \leq k} \mathcal{R}_{(2i)(2i-1)(2i-1)(2i)} g(\bar{Y}_{2i-1}, \bar{Y}_{2i-1})(g(\bar{X}_{2i}, \bar{X}_{2i}) + g(\bar{Y}_{2i}, \bar{Y}_{2i})) \\
&+ \sum_{i \leq k} \mathcal{R}_{(2i-1)(2i)(2i)(2i-1)} g(\bar{X}_{2i}, \bar{X}_{2i})(g(\bar{X}_{2i-1}, \bar{X}_{2i-1}) + g(\bar{Y}_{2i-1}, \bar{Y}_{2i-1})) \\
&+ \sum_{i \leq k} \mathcal{R}_{(2i-1)(2i)(2i)(2i-1)} g(\bar{Y}_{2i}, \bar{Y}_{2i})(g(\bar{X}_{2i-1}, \bar{X}_{2i-1}) + g(\bar{Y}_{2i-1}, \bar{Y}_{2i-1}))
\end{aligned}
$$

Note that every entry has a term $g(X_s, X_s) + g(Y_s, Y_s)$, which is equal to 0. Therefore, the scalar curvature vanishes, and one may use a similar set up to show $\rho^2$ and $|R|^2$ also vanish. $\square$

## 2.2 Curvature Homogeneity on $M_{f,h}$

*Proof of Theorem 1.5 (b). Case I: $f' \neq 0$.* We operate in the orthonormal frame $\beta_2$. Suppose that each of these entries is non-zero, and that all are equal to some function $a$. Note that because Equation (8) holds even if $i = j$, Equation (9) in conjunction with Equation (8) give us that

$$\lambda_{2i-1}\lambda_{2i} = \lambda_{2j-1}\lambda_{2j}$$

for all $i, j$. Then without loss of generality, we may proceed using only $\lambda_1$ and $\lambda_2$. Dividing the square of Equation (5) by Equation (8) gives us

$$a = \left(\frac{h''}{h'e^f}\right)^2. \tag{10}$$

Note that because $h', h'' \neq 0$, $a$ is both well-defined and positive. Dividing Equation (6) by Equation (8) yields

$$1 = -\frac{\sqrt{2}f'h'e^{2f}}{\lambda_2(h')^2e^f},$$

which implies

$$\lambda_2 = -\frac{\sqrt{2}f'e^f}{h'}.$$

However, we may also attain $\lambda_2$ by dividing Equation (4) by Equation (6). After simplification, we obtain

$$\lambda_2 = \frac{\sqrt{2}e^f((f')^2 + f'')}{f'h'}.$$

Both expressions must be valid, so we get that $f$ satisfies

$$f'' = -2(f')^2.$$

Now that we have $\lambda_2$, we may use it in conjuction with Equations (5) and (10) to find

$$\lambda_1 = \frac{\sqrt{2}h''e^f}{h'f'}.$$

For all other $\lambda_i$, they need only satisfy

$$\lambda_{2i-1}\lambda_{2i} = -\frac{2h''}{(h')^2}$$

to make all non-vanishing curvature entries equal to $\pm a$ at every point.

For some fixed $p \in M_{f,h}$, we may define a function $\varphi_p$ by

$$\varphi_p(q) = \frac{a(q)}{a(p)}.$$

We have then that

$$\mathcal{R}_{ijkl}(q) = \varphi_p(q)\mathcal{R}_{ijkl}(p)$$

for all $q \in M_{f,h}$. As $\varphi_p$ is smooth and positive and we are on an orthonormal frame, we have $M_{f,h}$ is CH(1,3) by Theorem 1.3.

*Case II: $f' = 0$.* If $f' = 0$, we have non-zero curvature entries given in $\beta_2$ by

$$\mathcal{R}_{(2i-1)00(2i)} = -\frac{\lambda_{2i-1}\lambda_{2i}}{2}\frac{h''}{e^{2f}}, \tag{11}$$

$$\mathcal{R}_{(2i-1)(2j)(2i)(2j-1)} = \frac{\lambda_{2i-1}\lambda_{2i}\lambda_{2j-1}\lambda_{2j}}{4}\frac{(h')^2}{e^{2f}}, \tag{12}$$

$$\mathcal{R}_{(2i-1)(2j-1)(2i)(2j)} = \frac{\lambda_{2i-1}\lambda_{2i}\lambda_{2j-1}\lambda_{2j}}{4}(1-\delta_{ij})\frac{(h')^2}{e^{2f}}. \tag{13}$$

Assume all non-zero entries are equal to some function $a$. We may use a similar trick from the first part to get that

$$\lambda_1\lambda_2 = \lambda_{2i-1}\lambda_{2i}.$$

We divide the square of Equation (11) by Equation (12) and get once again,

$$a(p) = \left(\frac{h''}{h'e^f}\right)^2.$$

Now we have that all non-zero entries are this function, provided

$$\lambda_{2i-1}\lambda_{2i} = -\frac{2h''}{(h')^2}.$$

We see that as we are still on the orthonormal frame $\beta_2$, we may define $\varphi$ in an identical manner so $M_{f,h}$ is CH(1,3).

$\square$

Consider the symmetric bilinear form $\phi_p$ on $T_pM$, which has non-zero entries given the following on the basis $\beta_1$.

$$\begin{aligned}
\phi_p(X_0, X_0) &= -1,\\
\phi_p(X_0, X_1) &= \epsilon(f),\\
\phi_p(X_0, Y_0) &= \epsilon(f),\\
\phi_p(X_{2i-1}, X_{2i}) &= 1,\\
\phi_p(X_{2i-1}, Y_{2i}) &= 1,\\
\phi_p(Y_{2i-1}, Y_{2i}) &= 1,
\end{aligned}$$

where $\epsilon(f) = 0$ if $f' = 0$, and is otherwise 1. After a few calculations, it is verified that the canonical algebraic curvature tensor $-R_{\sqrt{a(p)}\phi_p}$ is the Riemann curvature tensor at $p$.

*Proof of Theorem 1.5 (c).* If we have a scalar invariant of the 0-model which varies from point to point, the space will not be curvature homogeneous. Consider the trace of the form $\sqrt{a(p)}\phi_p$. This is not invariant to the space, as the curvature tensor is also defined by $-\sqrt{a(p)}\phi_p$. But by Theorem 1.4, these are the only canonical algebraic curvature tensors possible. Therefore, a quantity invariant to both forms is invariant to the space. The square of the trace is such a quantity.

We calculate the trace by looking at the self-adjoint operator $\sqrt{a(p)}\Phi_p$ associated to $\sqrt{a(p)}\phi_p$ in the orthonormal basis $\beta_2$. This is given by:

$$\sqrt{a}\begin{pmatrix}
-1 & \epsilon(f) & 0 & 0 & 0 & \dots & 0 & 0 & \epsilon(f) & 0 & 0 & 0 & \dots & 0 & 0\\
\epsilon(f) & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0\\
0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0\\
0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 1\\
0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0\\
-\epsilon(f) & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & -1 & 0 & 0 & \dots & 0 & 0\\
0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 & 0 & -1 & \dots & 0 & 0\\
0 & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 & -1 & 0 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & -1\\
0 & 0 & 0 & 0 & 0 & \dots & -1 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & 0
\end{pmatrix}.$$

It is easy to see then that the trace of $\pm\sqrt{a(p)}\Phi_p = \mp\sqrt{a(p)}$. This invariant is then equal to $a(p)$, which is constant for all $p$ if and only if $h' = c_0 h''$, where $c_0$ is a constant. Conversely, if $a(p)$ is constant for all $p$, then $R_{ijkl}(q) = R_{ijkl}(p)$, so $(M, g)$ is CH. □

## 2.3 Incompleteness of $M_{f,h}$

*Proof of Theorem 1.5 (d).* If $M_{f,h}$ is CH(1,3) and $f' \neq 0$, then $f$ is given by

$$f = \frac{1}{2}\log(2x_1 + c_1) + c_2,$$

where $c_1$ and $c_2$ are arbitrary constants. $M_{f,h}$ is defined only for $x_1 > -\frac{1}{2}c_1$. Consider the curve $\gamma(t) = (0, -\frac{1}{2}c_1(1-2t), 0, \ldots, 0)$. Differentiating, we get

$$\frac{d\gamma}{dt} = c_1 \partial_{x_1},$$

and it follows from Lemma 2.1 that

$$\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = (c_1)^2 \nabla_{\partial_{x_1}} \partial_{x_1} = 0.$$

$\gamma$ is therefore a geodesic. It is not defined for $t = 1$, so the space is not complete. □

# 3 The Geometry of $M_f$

**Lemma 3.1.** *For $M_f$,*

*(a) The non-zero Christoffel symbols are given by:*

$$\begin{aligned} \nabla_{\partial_{x_0}} \partial_{x_0} &= -f' e^{2f} \partial_{y_1}, \\ \nabla_{\partial_{x_0}} \partial_{x_1} &= f' \partial_{x_1}. \end{aligned}$$

*(b) Let $\Delta = (f')^2 + f''$. Up to the usual $\mathbb{Z}_2$ symmetries, the non-zero curvature entry of $\mathcal{R}$ is given by:*

$$\mathcal{R}_{1010} = e^{2f}\Delta,$$

*where all indices are of some $\partial_x$.*

*(c) Up to the usual $\mathbb{Z}_2$ symmetries, the non-zero entry of $\nabla^r \mathcal{R}$ is given by:*

$$\nabla^r \mathcal{R}_{1010;1\ldots1} = e^{2f}\Delta^{(r)}.$$

*Proof.* As in Lemma 2.1, we may use Equation 3 to calculate part (a), and Equations 1 and 2 to calculate part (b).

For part (c), we note that the non-zero curvature entry can only be differentiated by $\partial_{x_1}$, as differentiating by $\partial_{y_1}$ will yield zero outright by (a), and differentiating by $\partial_{x_0}$ must be zero because of the symmetries. The derivative of the exponential portion will cancel out when the $\partial_{x_0}$ entries are differentiated by $\partial_{x_1}$. For this manifold, the derivative of the zero entries of $\mathcal{R}$ and $\nabla^r \mathcal{R}$ is always zero. □

We construct two particularly useful moving frames on our manifold $\mathfrak{b}_1$ and $\mathfrak{b}_2$. Let $\mathfrak{b}_1$ be given by:

$$\begin{aligned} X_0 &= \frac{\partial_{x_0}}{e^f}, \\ X_1 &= \lambda \partial_{x_1}, \\ Y_1 &= \frac{1}{\lambda}\partial_{y_1}, \end{aligned}$$

where $\lambda$ is a smooth, non-vanishing function on $M_f$. $\mathfrak{b}_2$ is an orthonormal basis given by:

$$\begin{aligned} \bar{X}_0 &= X_0, \\ \bar{X}_1 &= \frac{X_1 + Y_1}{\sqrt{2}}, \\ \bar{Y}_1 &= \frac{X_1 - Y_1}{\sqrt{2}}. \end{aligned}$$

In $\mathfrak{b}_2$, the curvture entries are given by:

$$\nabla^r \mathcal{R}_{1010;1\ldots1} = \frac{\lambda^{r+2}}{2\sqrt{2}^r}\Delta^{(r)},$$

where the index 1 may be an index of either $\bar{X}_1$ or $\bar{Y}_1$.

*Proof of Theorem 1.6 (a).* We operate in the orthonormal frame $\mathfrak{b}_2$. As in the case of $M_{f,h}$, each entry with a $g(\bar{X}_1, \bar{X}_1)$ will be cancelled out by an entry with a $g(\bar{Y}_1, \bar{Y}_1)$. We calculate $\tau$ to demonstrate this.

In an orthonormal frame, we have

$$\begin{aligned} \tau &= \sum_{s,r} g^{ss} g^{rr} \mathcal{R}_{rssr} \\ &= g(\bar{X}_0, \bar{X}_0)\mathcal{R}_{1001}(g(\bar{X}_1, \bar{X}_1) + g(\bar{Y}_1, \bar{Y}_1)) \\ &\quad + g(\bar{X}_0, \bar{X}_0)\mathcal{R}_{0110}(g(\bar{X}_1, \bar{X}_1) + g(\bar{Y}_1, \bar{Y}_1)). \end{aligned}$$

Each term has $g(\bar{X}_1, \bar{X}_1) + g(\bar{Y}_1, \bar{Y}_1)$ in it, which is zero. Therefore, $\tau$ vanishes on $M_f$. In a similar method, one may show $|\rho|^2$ and $|R|^2$ also vanish. $\square$

*Proof of Theorem 1.6 (b).* We use the frame $\mathfrak{b}_2$. Fix some point $p \in M$, and define a function $\omega : M_f \to \mathbb{R}$ defined by

$$\omega(q) = \frac{\nabla^r \mathcal{R}_{1010;1\ldots1}(q)}{\nabla^r \mathcal{R}_{1010;1\ldots1}(p)}.$$

At any point $q \in M_f$, the non-zero curvature entries are given by $\omega(q)\nabla^r \mathcal{R}_{1010;1\ldots1}(p)$. We may do this independently for each $r$. By Theorem 1.3, $M_f$ is $\mathrm{CH}_r(1,3)$. $\square$

**Lemma 3.2.** *$M_f$ is weakly $CH_1$.*

*Proof.* Consider the frame $\mathfrak{b}_3 = \{\tilde{X}_0, \tilde{X}_1, \tilde{Y}_1\}$, where

$$\begin{aligned}\tilde{X}_0 &= X_0\\ \tilde{X}_1 &= \mu X_1\\ \tilde{Y}_1 &= Y_1,\end{aligned}$$

where $\mu = (\Delta')^{-1}$. In this frame, then, every non-zero curvature entry is given by $\pm 1$. The space is then weakly curvature homogeneous. □

Consider all bases which preserve this weak 1-model. Represented in $\mathfrak{b}_3$, these are bases obtained under the action of the matrix

$$A = \begin{pmatrix} \pm d^{-1} & b & 0 \\ 0 & d & 0 \\ s & t & c \end{pmatrix},$$

where $d, c$ are non-zero.

*Proof of Theorem 1.6 (c).* Suppose $M_f$ is weakly $\mathrm{CH}_2$. Then the function

$$\frac{\nabla^2 \mathcal{R}_{1010;11}}{(\nabla \mathcal{R}_{1010;1})^2}$$

must be constant in all frames of the form $A\mathfrak{b}_3$, as these are the frames where $\nabla \mathcal{R}_{1010;1}$ is constant, and if the space is weakly $\mathrm{CH}_n$, some frame must make all curvature entries constant. Thus this quantity may be expressed as

$$\frac{\nabla^2 \mathcal{R}_{1010;11}}{(\nabla \mathcal{R}_{1010;1})^2} = \frac{d^2 \Delta''}{d^2} = \Delta^2.$$

This can only be constant if $\Delta^2$ is constant. □

# 4 Further Work

The work done on this paper motivate a few new questions.

- Can the involved metric be generalized further and preserve CH(1,3)?
- Do all Weyl scalar invariants vanish on these manifolds?
- In general, the manifolds given are not complete. Are these manifolds non-extendible?
- Is it possible to have a space which is CH(1,3) of order 1, but not curvature homogeneous?
- Does $\mathrm{WCH}_n$ imply $\mathrm{CH}_{n+1}(1,3)$?

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