The Study of the Constant Vector Curvature Condition for Model Spaces in Dimension Three with Positive Definite Inner Product

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Abstract

We study the constant vector curvature condition in the simplest interesting case, for model spaces in dimension three with positive definite inner product.

Contents

1 Introduction

In differential geometry, one studies the geometric properties of differentiable manifolds. A *manifold* is a topological space which resembles Euclidean space locally. A metric on a manifold introduces *curvature* to the manifold. An example of a manifold is the surface of Earth: to an observer on Earth, the surface of Earth seems linear, but an observer in space can discern that the surface of Earth is curved. The utility of a manifold is that because it resembles Euclidean space locally, it has a coordinate system which enables one to do calculus in order to determine the curvature of the manifold. Johann Gauss proved the Theorema Egregium in 1827: an inhabitant of a two-dimensional surface can determine the curvature of the surface he or she inhabits without leaving the surface by measuring distances and angles. In 1854, Bernhard Riemann generalized the notion of curvature to manifolds of arbitrary finite dimension.
Curvature is one of the main properties of study in differential geometry, and as a result, there are many questions concerning curvature, including questions about different kinds of curvature. To answer a geometric question about a manifold, one can use linear algebra and calculus to answer an algebraic question about the tangent space to the manifold at a point, using a *model space*, which consists of a vector space, inner product, and algebraic curvature tensor. One crucial method for measuring the curvature of a model space is to measure the *sectional curvature* of the 2-planes. A model space which has the same value of sectional curvature for all of its 2-planes has *constant sectional curvature*. Constant sectional curvature is an important curvature condition because if a manifold has constant sectional curvature, then it is *homogeneous*, meaning there exists an isometry between any two of its points. Such manifolds are called *space forms*.
We study a new curvature condition which a model space can meet, the property of having *constant vector curvature*, introduced in 6. A model space has constant vector curvature ϵ if all of its nonzero vectors are contained in a nondegenerate 2-plane with sectional curvature ϵ. Since constant vector curvature is such a new condition, few things are known about the condition. Since constant vector curvature is a weaker condition than constant sectional curvature, in theory there should be model spaces which have constant vector curvature but do not have constant sectional curvature. Furthermore, by studying constant sectional curvature and constant vector curvature simultaneously, we can learn more about the structure of model spaces with each of the respective curvature conditions. We study the constant vector curvature condition in the simplest interesting case, for model spaces in dimension three with positive definite inner product and relate our findings to the eigenvalues of the associated Ricci tensor.
Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. Let ρ be the Ricci tensor, so

$$spec(\rho) = \{\lambda_1, \lambda_2, \lambda_3\}.$$

We prove the following theorems:

1. Constant vector curvature is well-defined.

2. $\mathcal{M}$ is Einstein if and only if $\mathcal{M}$ has $csc(\epsilon)$ if and only if $\lambda_1 = \lambda_2 = \lambda_3 = 2\epsilon$.

3. If $\mathcal{M}$ has $cvc(\epsilon)$, then for an orthonormal basis which diagonalizes ρ, the middle algebraic curvature tensor value of the possibly nonzero entries on the basis vectors is ϵ.

4. $\mathcal{M}$ has extremal constant vector curvature if and only if at least two of the eigenvalues of its Ricci tensor are the same. Specifically, if $\lambda_1 = \lambda_2$, then $\mathcal{M}$ has $ecvc(\frac{\lambda_3}{2})$.

5. If $\mathcal{M}$ has $cvc(0)$ and $||spec(\rho)|| = 3$, then $R = R_\phi \pm R_\psi$.

6. If $\mathcal{M}$ has $cvc(\epsilon)$, $R = R_\phi$, and $||spec(\rho)|| = 3$, then $\epsilon > 0$ and for an orthonormal basis which diagonalizes ρ, up to a change of sign, $\phi = diag\{\sqrt{\frac{\delta}{\tau}}, \sqrt{\delta\tau}, \sqrt{\frac{\tau}{\delta}}\}$, where τ and δ are the smallest and largest algebraic curvature tensor values of the possibly nonzero entries on the basis vectors, respectively.

7. There exists a model space such that $||spec(\rho)|| = 3$ which has constant vector curvature.

2 Definitions

In what follows, let V be a real finite dimensional vector space. We first define an algebraic curvature tensor.

Definition 1. *Let x, y, z, and w be in V. An* algebraic curvature tensor *is a function $R : V \times V \times V \times V \to \mathbb{R}$ such that*

1. *$R(x, y, z, w)$ is linear in every slot,*

2. *$R(x, y, z, w) = -R(y, x, z, w)$,*

3. *$R(x, y, z, w) = R(z, w, x, y)$, and*

4. *$R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$,*

where the fourth property is called the Bianchi Identity.

Definition 2. *Let R be an algebraic curvature tensor. The* kernel *of R is $\{x \in V | R(x, y, z, w) = 0 \ for\ all\ y \in V,\ for\ all\ z \in V,\ and\ for\ all\ w \in V\}$.*

The following is a sequence of the definitions of symmetric bilinear form, inner product, and positive definite inner product, which progress from more general to more specific.

Definition 3. *Let v and w be in V. A* symmetric bilinear form *is a function $\phi : V \times V \to \mathbb{R}$ such that*

1. *$\phi(v, w)$ is linear in every slot and*

2. $\phi(v, w) = \phi(w, v)$,

where the second property is called symmetry*.*

Definition 4. *An* inner product *is a symmetric bilinear form* $\langle \cdot, \cdot \rangle$ *which is* nondegenerate*, meaning that for all* v *in* V*,* $v \neq 0$*, there exists* w *in* V *such that* $\langle v, w \rangle \neq 0$*.*

Definition 5. *An inner product* $\langle \cdot, \cdot \rangle$ *is* positive definite *if for all* v *in* V *it is the case that* $\langle v, v \rangle > 0$ *if* $v \neq 0$ *and* $\langle v, v \rangle = 0$ *if and only if* $v = 0$*.*

We can now define a canonical algebraic curvature tensor, a special kind of algebraic curvature tensor built from a symmetric bilinear form.

Definition 6. *Let* x*,* y*,* z*, and* w *be in* V*. Let* ϕ *be a symmetric bilinear form. Define the* canonical algebraic curvature tensor $R_\phi(x, y, z, w) = \phi(x, w)\phi(y, z) - \phi(x, z)\phi(y, w)$*.*

Remark 1. R_ϕ *is an algebraic curvature tensor. See 3.*

Definition 7. *Let* $\langle \cdot, \cdot \rangle$ *be an inner product on* V*. Let* R *be an algebraic curvature tensor on* V*.* $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ *is a* model space*.*

In differential geometry, one studies properties of manifolds, such as curvature. A model space provides an algebraic portrait of the curvature of a manifold with a metric at a point by studying the tangent space of the manifold at said point.

Definition 8. *Let* $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ *be a model space. Let* v *and* w *be in* V*. Let* $\pi = span\{v, w\}$*.* π *is a* nondegenerate 2-plane *if* $\langle \cdot, \cdot \rangle|_\pi$ *is nondegenerate.*

Definition 9. *Let* $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ *be a model space. Let* v *and* w *be in* V*. Let* $\pi = span\{v, w\}$ *be a nondegenerate 2-plane. The* sectional curvature of π *is* $\kappa(\pi) = \frac{R(v,w,w,v)}{\langle v,v \rangle \langle w,w \rangle - \langle v,w \rangle \langle w,v \rangle}$*.*

We show that $\kappa(\pi)$ is well-defined in 4 and that the denominator is nonzero in 3.

Definition 10. *Let* $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ *be a model space.* $\mathcal{M}$ *has* constant sectional curvature ϵ *if for all nondegenerate 2-planes* π *it is the case that* $\kappa(\pi) = \epsilon$*. Denote constant sectional curvature* ϵ *by* $csc(\epsilon)$*.*

With the definition of constant sectional curvature established, we introduce the definition of constant vector curvature, first introduced in 6.

Definition 11. *Let* $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ *be a model space.* $\mathcal{M}$ *has* constant vector curvature ϵ *if for all* v *in* V*,* $v \neq 0$*, there exists* w *in* V *such that* $\kappa(span\{v, w\}) = \epsilon$*. Denote constant vector curvature* ϵ *by* $cvc(\epsilon)$*.*

The main difference between $csc(\epsilon)$ and $cvc(\epsilon)$ is that $csc(\epsilon)$ requires that for every vector in a vector space, every plane which contains said vector has sectional curvature ϵ, whereas $cvc(\epsilon)$ requires that for every vector in a vector space, there is some plane which contains said vector which has sectional curvature ϵ. Since $csc(\epsilon)$ has a universal quantifier followed by another universal quantifier while $cvc(\epsilon)$ has a universal quantifier followed by an existential quantifier, $csc(\epsilon)$ implies $cvc(\epsilon)$.

Definition 12. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. $\mathcal{M}$ has* extremal constant vector curvature ϵ *if $\mathcal{M}$ has $cvc(\epsilon)$ and for all nondegenerate 2-planes π it is the case that either $\kappa(\pi) \leq \epsilon$ or $\kappa(\pi) \geq \epsilon$. Denote extremal constant vector curvature ϵ by $ecvc(\epsilon)$.*

Remark 2. *A manifold with an indefinite metric which has a bound on the sectional curvature of its 2-planes has $csc(\epsilon)$. See 5.*

Definition 13. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. Let v and w be in V. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. The* Ricci tensor *is the symmetric bilinear form ρ defined by*

$$\rho(v, w) = \sum_{i=1}^{n} \langle e_i, e_i\rangle R(v, e_i, e_i, w).$$

ρ is independent of the orthonormal basis chosen. See 3.

Definition 14. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. $\mathcal{M}$ is* Einstein *if $\rho = c\langle\cdot,\cdot\rangle$ for some constant c.*

Definition 15. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle)$ be a model space. Let $A : V \to V$ be a linear function. The* adjoint *of A is the linear map $A^* : V \to V$ such that for all v in V and for all w in V it is the case that $\langle A^*v, w\rangle = \langle v, Aw\rangle$.*

3 Lemmas

Lemma 1. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space. For all x in V, for all v in V, for all w in V, and for all z in V it is the case that $R(x, x, v, w) = 0$ and $R(x, v, w, z) = R(v, x, z, w)$.*

Proof. By the second property of the definition of algebraic curvature tensor, $R(x, x, v, w) = -R(x, x, v, w)$. Hence, $R(x, x, v, w) = 0$. By the second and third properties of the definition of algebraic curvature tensor,

$$\begin{aligned} R(x, v, w, z) &= -R(v, x, w, z) \\ &= -R(w, z, v, x) \\ &= -(-R(z, w, v, x)) \\ &= R(z, w, v, x) \\ &= R(v, x, z, w). \end{aligned}$$

□

Lemma 2. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. Let v and w be in V. Let $\pi = span\{v, w\}$ be a nondegenerate 2-plane. Let $\{e_1, e_2\}$ and $\{w, v\}$ be bases for π. Let $A : span\{e_1, e_2\} \to span\{v, w\}$ be a linear function which is one-to-one and onto. Let a, b, c, and d be constants such that*

$$\begin{aligned} Ae_1 = v = ae_1 + be_2, \\ Ae_2 = w = ce_1 + de_2. \end{aligned}$$

Then, $\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle w, v\rangle = (ad - bc)^2(\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle\langle e_2, e_1\rangle)$.

Proof.

$$\begin{aligned} \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle w, v\rangle &= \langle ae_1 + be_2, ae_1 + be_2\rangle\langle ce_1 + de_2, ce_1 + de_2\rangle - \\ &\quad \langle ae_1 + be_2, ce_1 + de_2\rangle\langle ce_1 + de_2, ae_1 + be_2\rangle \\ &= [(\langle ae_1, ae_1\rangle + \langle ae_1, be_2\rangle + \langle be_2, ae_1\rangle + \langle be_2, be_2\rangle) \\ &\quad (\langle ce_1, ce_1\rangle + \langle ce_1, de_2\rangle + \langle de_2, ce_1\rangle + \langle de_2, de_2\rangle)] - \\ &\quad [(\langle ae_1, ce_1\rangle + \langle ae_1, de_2\rangle + \langle be_2, ce_1\rangle + \langle be_2, de_2\rangle) \\ &\quad (\langle ce_1, ae_1\rangle + \langle ce_1, be_2\rangle + \langle de_2, ae_1\rangle + \langle de_2, be_2\rangle)] \\ &= (a^2d^2 - 2abcd + b^2c^2)(\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle\langle e_2, e_1\rangle) \\ &= (ad - bc)^2(\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle\langle e_2, e_1\rangle). \end{aligned}$$

□

Corollary 1. *For A as in 2, $ad - bc \neq 0$.*

Proof. Since $rank(A) = 2$, $ker(A) = \{0\}$, so $det(A) = ad - bc \neq 0$. □

By definition, a 2-plane π is nondegenerate if and only if the inner product $\langle \cdot, \cdot \rangle|_\pi$ is nondegenerate. If $\langle \cdot, \cdot \rangle|_\pi$ is nondegenerate, then there exists an orthonormal basis for pi. In the nondegenerate setting, each orthonormal basis vector either has length 1, in which case the basis vector is spacelike, or -1, in which case the basis vector is timelike. This is different from the positive definite setting, where each orthonormal basis vector has length 1.

We prove that the denominator of sectional curvature is nonzero.

Lemma 3. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. Let v and w be vectors in V. If $\pi = span\{v, w\}$ is a nondegenerate 2-plane, then $\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle w, v\rangle \neq 0$.*

Proof. Assume $\pi = span\{v, w\}$ is a nondegenerate 2-plane. Let $\{e_1, e_2\}$ be a basis for π which is orthonormal with respect to $\langle \cdot, \cdot \rangle$. Let A be as in 2. So,

$$\begin{aligned} \langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle\langle w, v\rangle &= (ad - bc)^2(\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle\langle e_2, e_1\rangle). \\ &= (ad - bc)^2[(\pm 1)(\pm 1) - (0)(0)] \\ &= \pm(ad - bc)^2 \\ &\neq 0. \end{aligned}$$

□

We prove that sectional curvature is well-defined.

Lemma 4. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ be a model space. Let $\{e_1, e_2\}$, $\{f_1, f_2\}$ be bases for nondegenerate 2-planes. If $\pi = span\{e_1, e_2\} = span\{f_1, f_2\}$, then $\kappa(span\{e_1, e_2\}) = \kappa(span\{f_1, f_2\})$.*

Proof. Assume $\pi = span\{e_1, e_2\} = span\{f_1, f_2\}$. Let A be as in 2. We evaluate $\frac{R(f_1,f_2,f_2,f_1)}{\langle f_1,f_1\rangle\langle f_2,f_2\rangle - \langle f_1,f_2\rangle\langle f_2,f_1\rangle}$, computing the numerator and denominator separately.
Numerator:

$$\begin{aligned} R(f_1, f_2, f_2, f_1) &= R(ae_1 + be_2, ce_1 + de_2, ce_1 + de_2, ae_1 + be_2) \\ &= R(ae_1, ce_1, ce_1, ae_1) + R(ae_1, ce_1, ce_1, be_2) + R(ae_1, ce_1, de_2, ae_1) + \\ &\quad R(ae_1, ce_1, de_2, be_2) + R(ae_1, de_2, ce_1, ae_1) + R(ae_1, de_2, ce_1, be_2) + \\ &\quad R(ae_1, de_2, de_2, ae_1) + R(ae_1, de_2, de_2, be_2) + R(be_2, ce_1, ce_1, ae_1) + \\ &\quad R(be_2, ce_1, ce_1, be_2) + R(be_2, ce_1, de_2, ae_1) + R(be_2, ce_1, de_2, be_2) + \\ &\quad R(be_2, de_2, ce_1, ae_1) + R(be_2, de_2, ce_1, be_2) + R(be_2, de_2, de_2, ae_1) + \\ &\quad R(be_2, de_2, de_2, be_2) \\ &= a^2d^2R(e_1, e_2, e_2, e_1) - 2abcdR(e_1, e_2, e_2, e_1) + b^2c^2R(e_1, e_2, e_2, e_1) \\ &= (ad - bc)^2R(e_1, e_2, e_2, e_1). \end{aligned}$$

Denominator:

$$\langle f_1, f_1\rangle\langle f_2, f_2\rangle - \langle f_1, f_2\rangle\langle f_2, f_1\rangle = (ad - bc)^2(\langle e_1, e_1\rangle\langle e_2, e_2\rangle - \langle e_1, e_2\rangle\langle e_2, e_1\rangle).$$

Hence,

$$\begin{aligned} \frac{R(f_1,f_2,f_2,f_1)}{\langle f_1,f_1\rangle\langle f_2,f_2\rangle - \langle f_1,f_2\rangle\langle f_2,f_1\rangle} &= \frac{(ad-bc)^2R(e_1,e_2,e_2,e_1)}{(ad-bc)^2(\langle e_1,e_1\rangle\langle e_2,e_2\rangle - \langle e_1,e_2\rangle\langle e_2,e_1\rangle)} \\ &= \frac{R(e_1,e_2,e_2,e_1)}{\langle e_1,e_1\rangle\langle e_2,e_2\rangle - \langle e_1,e_2\rangle\langle e_2,e_1\rangle}. \end{aligned}$$

Therefore, $\kappa(span\{e_1, e_2\}) = \kappa(span\{f_1, f_2\})$. □

Lemma 5. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$. Let v be in V, $v \neq 0$, and w be in V, $w \neq 0$, such that $\langle v, v\rangle = 1$, $\langle w, w\rangle = 1$, and $\langle v, w\rangle = 0$. Then, $\kappa(span\{v, w\}) = R(v, w, w, v)$.*

Proof.

$$\begin{aligned} \kappa(span\{v, w\}) &= \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\ &= \frac{R(v,w,w,v)}{(1)(1)-(0)(0)} \\ &= R(v, w, w, v). \end{aligned}$$

□

We prove a lemma which enables constant vector curvature to be classified into three different general cases.

Lemma 6. *Assume $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$ has cvc(ϵ). For some $\tilde{\langle \cdot, \cdot \rangle} = c\langle \cdot, \cdot \rangle$, where c is some constant, $\tilde{\mathcal{M}} = (V, \tilde{\langle \cdot, \cdot \rangle}, R)$ has cvc(δ), where either $\delta = -1$, $\delta = 0$, or $\delta = 1$.*

Proof. Assume $\mathcal{M}$ has $cvc(\epsilon)$. If $\epsilon = 0$, then choose $\langle \tilde{\cdot, \cdot} \rangle = \langle \cdot, \cdot \rangle$. So, $\tilde{\mathcal{M}} = \mathcal{M}$, which has $cvc(0)$. If $\epsilon \neq 0$, then choose $\langle \tilde{\cdot, \cdot} \rangle = |\epsilon|^{\frac{1}{2}} \langle \cdot, \cdot \rangle$.
Since $\mathcal{M}$ has $cvc(\epsilon)$, for all v in V, $v \neq 0$, there exists w in V such that

$$\frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} = \epsilon.$$

But,

$$\begin{aligned} \frac{R(v,w,w,v)}{\langle \tilde{v,v}\rangle\langle \tilde{w,w}\rangle - \langle \tilde{v,w}\rangle\langle \tilde{w,v}\rangle} &= \frac{R(v,w,w,v)}{|\epsilon|^{\frac{1}{2}}\langle v,v\rangle|\epsilon|^{\frac{1}{2}}\langle w,w\rangle - |\epsilon|^{\frac{1}{2}}\langle v,w\rangle|\epsilon|^{\frac{1}{2}}\langle w,v\rangle} \\ &= \frac{R(v,w,w,v)}{|\epsilon|(\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle)} \\ &= \frac{\epsilon}{|\epsilon|}. \end{aligned}$$

If $\epsilon < 0$, then $\frac{\epsilon}{|\epsilon|} = -1$. If $\epsilon > 0$, then $\frac{\epsilon}{|\epsilon|} = 1$. Since v was chosen arbitrarily, $\tilde{\mathcal{M}}$ has either $cvc(-1)$ or $cvc(1)$. Therefore, $\tilde{\mathcal{M}} = (V, \langle \tilde{\cdot, \cdot} \rangle, R)$ has $cvc(\delta)$, where either $\delta = -1$, $\delta = 0$, or $\delta = 1$. □

Lemma 7. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$. If $\mathcal{M}$ has $cvc(\epsilon)$ and $ker(R) \neq \{0\}$, then $\epsilon = 0$.*

Proof. Let v be in $ker(R)$, $v \neq 0$. Then, for all w in V,

$$\begin{aligned} \kappa(span\{v,w\}) &= \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\ &= 0. \end{aligned}$$

So, every nondegenerate 2-plane which contains v has sectional curvature 0. Since $\mathcal{M}$ has $cvc(\epsilon)$, some nondegenerate 2-plane which contains v must have sectional curvature ϵ. Therefore, $\epsilon = 0$. □

We introduce a series of lemmas from linear algebra which will be instrumental in proving subsequent lemmas and theorems.

Lemma 8. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle)$, where $\langle \cdot, \cdot \rangle$ is positive definite. Let $A : V \to V$ be a linear function. If $A = A^*$, then there exists a basis $\{e_1, \ldots, e_n\}$ for V which is orthonormal with respect to $\langle \cdot, \cdot \rangle$ such that*

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix},$$

where $\langle e_i, e_i \rangle = \lambda_i$.

Proof. See 2. □

Lemma 9. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle)$ be a model space, where $\langle \cdot, \cdot \rangle$ is nondegenerate. Let ϕ be a symmetric bilinear form. Then, there exists a unique linear function $A : V \to V$ such that for all v in V and for all w in V it is the case that $\phi(v, w) = \langle Av, w \rangle$.*

Proof. See 2. ☐

Lemma 10. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is positive definite. Let ϕ be a symmetric bilinear form. Let $A : V \to V$ be the linear function such that for all v and for all w it is the case that $\phi(v, w) = \langle Av, w\rangle$. Then, there exists a basis $\{e_1, \ldots, e_n\}$ for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that $\phi(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$.*

Proof. We prove the well-known claim, which is proved in 2. Since $\phi(v, w) = \langle Av, w\rangle$, we see that

$$\begin{aligned}\langle Av, w\rangle &= \phi(v, w)\\ &= \phi(w, v)\\ &= \langle Aw, v\rangle\\ &= \langle v, Aw\rangle.\end{aligned}$$

So, since $\langle\cdot,\cdot\rangle$ is positive definite, $A = A^*$. A is as in 8, so there exists a basis $\{e_1, \ldots, e_n\}$ for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that

$$\begin{aligned}\phi(e_i, e_j) &= \langle Ae_i, e_j\rangle\\ &= \langle \lambda_i e_i, e_j\rangle\\ &= \langle e_i, e_j\rangle\lambda_i.\end{aligned}$$

☐

Corollary 2. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$ be a model space, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. Then, there exists a basis $\{e_1, e_2, e_3\}$ for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that $R(e_i, e_j, e_k, e_i) = 0$ when $i \neq j$, $i \neq k$, and $j \neq k$.*

Proof. Let A and $\{e_1, e_2, e_3\}$ be as in 10. So,

$$\begin{aligned}R(e_i, e_j, e_k, e_i) &= R(e_j, e_i, e_i, e_k) + R(e_j, e_j, e_j, e_k) + R(e_j, e_k, e_k, e_k)\\ &= \rho(e_j, e_k)\\ &= \langle e_j, e_k\rangle\lambda_j\\ &= 0.\end{aligned}$$

☐

The following lemma relates constant sectional curvature and algebraic curvature tensors.

Lemma 11. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$. $R = \epsilon R_{\langle\cdot,\cdot\rangle}$ if and only if $\mathcal{M}$ has $csc(\epsilon)$.*

Proof. See 3. ☐

Lemma 12. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. $R = R_\phi \pm R\psi$ if and only if, without loss of generality, $\lambda_i \neq 0$, $\lambda_j \neq 0$, and $\lambda_k = \lambda_i + \lambda_j$.*

Proof. See 1. ☐

We prove a series of lemmas which relate the Ricci tensor to algebraic curvature tensors.

Lemma 13. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$. Then, $R(e_i, e_j, e_j, e_i) = \frac{\lambda_i+\lambda_j-\lambda_k}{2}$.*

Proof.

$$\begin{aligned} R(e_i, e_j, e_j, e_i) &= \tfrac{R(e_i,e_i,e_i,e_i)+R(e_i,e_j,e_j,e_i)+R(e_i,e_k,e_k,e_i)}{2} + \\ &\quad \tfrac{R(e_j,e_i,e_i,e_j)+R(e_j,e_j,e_j,e_j)R(e_j,e_k,e_k,e_j)}{2} - \\ &\quad \tfrac{R(e_k,e_i,e_i,e_k)+R(e_k,e_j,e_j,e_k)+R(e_k,e_k,e_k,e_k)}{2} \\ &= \tfrac{\rho(e_i,e_i)+\rho(e_j,e_j)-\rho(e_k,e_k)}{2} \\ &= \tfrac{\lambda_i+\lambda_j-\lambda_k}{2}. \end{aligned}$$

□

Lemma 14. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$. Then,*

$$||\{R(e_1, e_2, e_2, e_1), R(e_1, e_3, e_3, e_3), R(e_2, e_3, e_3, e_2)\}|| = ||spec(\rho)||.$$

Proof. Assume

$$||\{R(e_1, e_2, e_2, e_1), R(e_1, e_3, e_3, e_3), R(e_2, e_3, e_3, e_2)\}|| = 1.$$

This is the case if and only if

$$R(e_i, e_j, e_j, e_i) = R(e_i, e_k, e_k, e_i) = R(e_j, e_k, e_k, e_j).$$

By 13,

$$\begin{aligned} \tfrac{\lambda_i+\lambda_j-\lambda_k}{2} &= R(e_i, e_j, e_j, e_i) \\ &= R(e_i, e_k, e_k, e_i) \\ &= \tfrac{\lambda_i+\lambda_j-\lambda_k}{2}. \end{aligned}$$

Since the denominators of both expressions are the same,

$$\begin{aligned} \lambda_i + \lambda_j - \lambda_k &= \lambda_i + \lambda_k - \lambda_j, \text{ which is the case if and only if} \\ \lambda_j - \lambda_k &= \lambda_k - \lambda_j, \text{ which is the case if and only if} \\ 2\lambda_j - 2\lambda_k &= 0, \text{ which is the case if and only if} \\ \lambda_j - \lambda_k &= 0, \text{ which is the case if and only if} \\ \lambda_j &= \lambda_k. \end{aligned}$$

By analogous reasoning, $\lambda_i = \lambda_k$, so $\lambda_i = \lambda_j = \lambda_k$. Hence,

$$||\{R(e_1, e_2, e_2, e_1), R(e_1, e_3, e_3, e_3), R(e_2, e_3, e_3, e_2)\}|| = 1$$

if and only if $||spec(\rho)|| = 1$.
Assume

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_3), R(e_2,e_3,e_3,e_2)\}|| = 2.$$

Assume without loss of generality that

$$R(e_1,e_2,e_2,e_1) = R(e_1,e_3,e_3,e_1) \neq R(e_2,e_3,e_3,e_2).$$

This is the case if and only if $\lambda_2 = \lambda_3$. Suppose $\lambda_1 = \lambda_2$. This is the case if and only if

$$\begin{aligned} R(e_2,e_3,e_3,e_2) &= \tfrac{\lambda_2+\lambda_3-\lambda_1}{2} \\ &= \tfrac{\lambda_1+\lambda_3-\lambda_2}{2} \\ &= R(e_1,e_3,e_3,e_1). \end{aligned}$$

This contradicts the assumption that $R(e_1,e_3,e_3,e_1) \neq R(e_2,e_3,e_3,e_2)$. So, $\lambda_1 \neq \lambda_2$, so $\lambda_1 \neq \lambda_2 = \lambda_3$. Hence,

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_3), R(e_2,e_3,e_3,e_2)\}|| = 2$$

if and only if $||spec(\rho)|| = 2$.
Assume

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_3), R(e_2,e_3,e_3,e_2)\}|| = 3.$$

Without loss of generality, suppose $\lambda_1 = \lambda_2$. This is the case if and only if

$$\begin{aligned} R(e_1,e_3,e_3,e_1) &= \tfrac{\lambda_1+\lambda_3-\lambda_2}{2} \\ &= \tfrac{\lambda_2+\lambda_3-\lambda_1}{2} \\ &= R(e_2,e_3,e_3,e_2). \end{aligned}$$

This contradicts the assumption that

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_3), R(e_2,e_3,e_3,e_2)\}|| = 3.$$

So, $\lambda_1 \neq \lambda_2$. By analogous reasoning, $\lambda_1 \neq \lambda_3$ and $\lambda_2 \neq \lambda_3$. Hence,

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_3), R(e_2,e_3,e_3,e_2)\}|| = 3$$

if and only if $||spec(\rho)|| = 3$.

□

The following lemma relates the geometry of a three-dimensional vector space to the sectional curvature of its 2-planes, introduced in 4.

Lemma 15. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $\langle\cdot,\cdot\rangle$ is positive definite. Let v be in V, w be in V, and u be in V such that $\langle v,v\rangle = 1$, $\langle w,w\rangle = 1$, $\langle u,u\rangle = 1$, $\langle v,w\rangle = 0$, $\langle v,u\rangle = 0$, and $\langle w,u\rangle = 0$. Let $w_\theta = \cos\theta w + \sin\theta u$. Then,*

$$\kappa(span\{v,w_\theta\}) = \cos^2\theta R(v,w,w,v) + \sin^2\theta R(v,u,u,v) + 2\cos\theta\sin\theta R(v,w,u,v).$$

Proof.

$$
\begin{aligned}
\kappa(span\{v, w_\theta\}) &= \frac{R(v,w_\theta,w_\theta,v)}{\langle v,v\rangle\langle w_\theta,w_\theta\rangle - \langle v,w_\theta\rangle\langle w_\theta,v\rangle} \\
&= \frac{R(v,\cos\theta w+\sin\theta u,\cos\theta w+\sin\theta u,v)}{\langle v,v\rangle\langle \cos\theta w+\sin\theta u,\cos\theta w+\sin\theta u\rangle - \langle v,\cos\theta w+\sin\theta u\rangle\langle \cos\theta w+\sin\theta u,v\rangle} \\
&= \frac{R(v,\cos\theta w,\cos\theta w,v)+R(v,\cos\theta w,\sin\theta u,v)+R(v,\sin\theta u,\cos\theta w,v)+R(v,\sin\theta u,\sin\theta u,v)}{(\langle v,v\rangle)(\langle\cos\theta w,\cos\theta w\rangle+\langle\cos\theta w,\sin\theta u\rangle+\langle\sin\theta u,\cos\theta w\rangle+\langle\sin\theta u,\sin\theta u\rangle)} - \\
&\quad \frac{R(v,\cos\theta w,\cos\theta w,v)+R(v,\cos\theta w,\sin\theta u,v)+R(v,\sin\theta u,\cos\theta w,v)+R(v,\sin\theta u,\sin\theta u,v)}{(\langle v,\cos\theta w\rangle+\langle v,\sin\theta u\rangle)(\langle\cos\theta w,v\rangle+\langle\sin\theta u,v\rangle)} \\
&= \frac{\cos^2\theta R(v,w,w,v)+2\cos\theta\sin\theta R(v,w,u,v)+\sin^2\theta R(v,u,u,v)}{(\langle v,v\rangle)(\cos^2\theta\langle w,w\rangle+2\cos\theta\sin\theta\langle w,u\rangle+sin^2\theta\langle u,u\rangle)-(\cos\theta\langle v,w\rangle+\sin\theta\langle v,u\rangle)(\cos\theta\langle w,v\rangle+\sin\theta\langle u,v\rangle)} \\
&= \frac{\cos^2\theta R(v,w,w,v)+2\cos\theta\sin\theta R(v,w,u,v)+\sin^2\theta R(v,u,u,v)}{(1)(\cos^2\theta(1)+\sin^2\theta(1))-(\cos\theta(0)+\sin\theta(0))(\cos\theta(0)+\sin\theta(0))} \\
&= \frac{\cos^2\theta R(v,w,w,v)+2\cos\theta\sin\theta R(v,w,u,v)+\sin^2\theta R(v,u,u,v)}{(1)(1)} \\
&= \cos^2\theta R(v,w,w,v) + 2\cos\theta\sin\theta R(v,w,u,v) + \sin^2\theta R(v,u,u,v).
\end{aligned}
$$

□

Lemma 16. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. If $\lambda_1 = \lambda_2$, then $\mathcal{M}$ has $cvc(\frac{\lambda_3}{2})$.*

Proof. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$. Assume $\lambda_1 = \lambda_2$. Then,

$$
\begin{aligned}
R(e_1, e_3, e_3, e_1) &= \frac{\lambda_1+\lambda_3-\lambda_2}{2} \\
&= \frac{\lambda_1+\lambda_3-\lambda_1}{2} \\
&= \frac{\lambda_3}{2}.
\end{aligned}
$$

By analogous reasoning, $R(e_2, e_3, e_3, e_2) = \frac{\lambda_3}{2}$. Let $w_\theta = \cos\theta e_1 + \sin\theta e_2$. Then, by 15, let $f(\theta) =$

$$
\begin{aligned}
\kappa(span\{e_3, w_\theta\}) &= \cos^2\theta R(e_1, e_3, e_3, e_1) + \sin^2\theta R(e_2, e_3, e_3, e_2) + 2\cos\theta\sin\theta R(e_3, e_1, e_2, e_3) \\
&= \cos^2\theta\frac{\lambda_3}{2} + \sin^2\theta\frac{\lambda_3}{2} \\
&= \frac{\lambda_3}{2}.
\end{aligned}
$$

Since $f(\theta)$ for values of θ from 0 to π represents every 2-plane which contains e_3, every nondegenerate 2-plane which contains e_3 has sectional curvature $\frac{\lambda_3}{2}$. Let v be in V, $v \neq 0$. If either $v = -e_3$ or $v = e_3$, then choose any w in V, $w \neq 0$, $w \neq -e_3$, $w \neq e_3$. Without loss of generality, assume $\langle v, v\rangle = 1$. Since there exists some w' such that $\langle w', w'\rangle = 1$, $\langle v, w'\rangle = 0$, $span\{v, w\} = span\{v, w'\}$, and $\kappa(span\{v, w'\}) = \frac{\lambda_3}{2}$, $\kappa(span\{v, w\}) = \frac{\lambda_3}{2}$. If $v \neq -e_3$ and $v \neq e_3$, then choose $w = e_3$. Since there exists some v' such that $\langle v', v'\rangle = 1$, $\langle w, w\rangle = 1$, $\langle v', w\rangle = 0$, $span\{v, w\} = span\{v', w\}$, and $\kappa(span\{v', w\}) = \frac{\lambda_3}{2}$, $\kappa(span\{v, w\}) = \frac{\lambda_3}{2}$. So, for all v in V, $v \neq 0$, there exists w in V such that $\kappa(span\{v, w\}) = \frac{\lambda_3}{2}$. Therefore, $\mathcal{M}$ has $cvc(\frac{\lambda_3}{2})$. □

We introduce a series of lemmas which relate orthonormal bases to canonical algebraic curvature tensors.

Lemma 17. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $R = R_\phi$. There exists a basis $\{e_1, e_2, e_3\}$ for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ.*

Proof. See 3. ☐

Lemma 18. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $R = R_\phi$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ. Then, $R(e_i, e_j, e_j, e_i) = \eta_i\eta_j$, where $\phi(e_i, e_i) = \eta_i$.*

Proof.

$$\begin{aligned} R(e_i, e_j, e_j, e_i) &= R_\phi(e_i, e_j, e_j, e_i) \\ &= \phi(e_i, e_i)\phi(e_j, e_j) - \phi(e_i, e_j)\phi(e_j, e_i) \\ &= \eta_i\eta_j. \end{aligned}$$

☐

Lemma 19. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$, $\langle\cdot,\cdot\rangle$ is positive definite, and $R = R_\phi$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ. If $\mathcal{M}$ has $cvc(\epsilon)$ and*

$$||\{R(e_1, e_2, e_2, e_1), R(e_1, e_3, e_3, e_1), R(e_2, e_3, e_3, e_2)\}|| = 3,$$

then either $\eta_1 > \eta_2 > \eta_3 > 0$ or $0 > \eta_3 > \eta_2 > \eta_1$.

Proof. By 7, $\epsilon > 0$. Without loss of generality assume

$$R(e_1, e_2, e_2, e_1) = \eta_1\eta_2 > R(e_1, e_3, e_3, e_1) = \eta_1\eta_3 > R(e_2, e_3, e_3, e_2) = \eta_2\eta_3.$$

By 4, $R(e_1, e_3, e_3, e_1) = \epsilon$. By 6, without loss of generality, assume $\epsilon = 1$. Since $R(e_1, e_3, e_3, e_1) = \eta_1\eta_3 = 1$, either both $\eta_1 > 0$ and $\eta_3 > 0$ or both $\eta_1 < 0$ and $\eta_3 < 0$. Suppose $\eta_1 > 0$ and $\eta_3 > 0$. Since $\eta_1\eta_2 > \eta_1\eta_3$, $\eta_2 > 0$. Since $\eta_1\eta_2 > \eta_1\eta_3 > \eta_2\eta_3$, $\eta_1 > \eta_2 > \eta_3$. So, $\eta_1 > \eta_2 > \eta_3 > 0$. Suppose $\eta_1 < 0$ and $\eta_3 < 0$. Since $\eta_1\eta_2 > \eta_1\eta_3$, $\eta_2 < 0$. Since $\eta_1\eta_2 > \eta_1\eta_3 > \eta_2\eta_3$, $\eta_3 > \eta_2 > \eta_1$. So, $0 > \eta_3 > \eta_2 > \eta_1$. ☐

Lemma 20. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. Then,*

$$\begin{aligned} \lambda_1 &= \delta + \sigma, \\ \lambda_2 &= \delta + \tau, \text{and} \\ \lambda_3 &= \sigma + \tau \end{aligned}$$

where $R(e_1, e_2, e_2, e_1) = \delta$, $R(e_1, e_3, e_3, e_1) = \sigma$, and $R(e_2, e_3, e_3, e_2) = \tau$.

Proof. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to

$\langle \cdot, \cdot \rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j \rangle \lambda_i$.

$$\begin{aligned}
\lambda_1 &= \tfrac{2\lambda_1}{2} \\
&= \tfrac{\lambda_1+\lambda_2-\lambda_3+\lambda_1+\lambda_3-\lambda_2}{2} \\
&= \tfrac{\lambda_1+\lambda_2-\lambda_3}{2} + \tfrac{\lambda_1+\lambda_3-\lambda_2}{2} \\
&= R(e_1, e_2, e_2, e_1) + R(e_1, e_3, e_3, e_1) \\
&= \delta + \sigma. \\
\lambda_2 &= \tfrac{2\lambda_2}{2} \\
&= \tfrac{\lambda_2+\lambda_1-\lambda_3+\lambda_2+\lambda_3-\lambda_1}{2} \\
&= \tfrac{\lambda_2+\lambda_1-\lambda_3}{2} + \tfrac{\lambda_2+\lambda_3-\lambda_1}{2} \\
&= R(e_1, e_2, e_2, e_1) + R(e_2, e_3, e_3, e_2) \\
&= \delta + \tau. \\
\lambda_3 &= \tfrac{2\lambda_3}{2} \\
&= \tfrac{\lambda_3+\lambda_1-\lambda_2+\lambda_3+\lambda_2-\lambda_1}{2} \\
&= \tfrac{\lambda_3+\lambda_1-\lambda_2}{2} + \tfrac{\lambda_3+\lambda_2-\lambda_1}{2} \\
&= R(e_1, e_3, e_3, e_1) + R(e_2, e_3, e_3, e_2) \\
&= \sigma + \tau.
\end{aligned}$$

□

4 Theorems

Theorem 1. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, where $dim(V) = 3$ and $\langle \cdot, \cdot \rangle$ is positive definite. If $\mathcal{M}$ has $cvc(\epsilon)$ and $cvc(\delta)$, then $\epsilon = \delta$.*

Proof. Assume $\mathcal{M}$ has $cvc(\epsilon)$ and $cvc(\delta)$. By 10, let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle \cdot, \cdot \rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j \rangle \lambda_i$. Let $v = e_1$.
Since $\mathcal{M}$ has $cvc(\epsilon)$, there exists w in V such that

$$\frac{R(v, w, w, v)}{\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle \langle w, v \rangle} = \epsilon.$$

Since $\langle \cdot, \cdot \rangle$ is positive definite, every 2-plane is nondegenerate. Since $span\{v, w\}$ is a nondegenerate 2-plane, there exists w' in V such that $span\{v, w\} = span\{v, w'\}$, where w' is orthogonal to v, by the Gram-Schmidt process. By [Lemma 2], $\kappa(span\{v, w\}) = \kappa(span\{v, w'\})$. Without loss of generality, assume $\langle v, w \rangle = 0$ and $\langle w, w \rangle = 1$.
Since $\mathcal{M}$ has $cvc(\delta)$, there exists u in V such that

$$\frac{R(v, u, u, v)}{\langle v, v \rangle \langle u, u \rangle - \langle v, u \rangle \langle u, v \rangle} = \delta.$$

By analogous reasoning, assume $\langle v, u \rangle = 0$ and $\langle u, u \rangle = 1$.
Since $\mathcal{M}$ has $cvc(\delta)$, there exists x in V such that

$$\frac{R(w, x, x, w)}{\langle w, w \rangle \langle x, x \rangle - \langle w, x \rangle \langle x, w \rangle} = \delta.$$

By analogous reasoning, assume $\langle w, x \rangle = 0$ and $\langle x, x \rangle = 1$.
Since $\mathcal{M}$ has $cvc(\epsilon)$, there exists y in V such that

$$\frac{R(u,y,y,u)}{\langle u,u\rangle\langle y,y\rangle - \langle u,y\rangle\langle y,u\rangle} = \epsilon.$$

By analogous reasoning, assume $\langle u, y \rangle = 0$ and $\langle y, y \rangle = 1$.
Let a, b, c, d, f, g, h, and j be constants such that

$$\begin{gathered} v = e_1, \\ w = ae_2 + be_3, \\ u = ce_2 + de_3, \\ x = fe_1 + g(be_2 - ae_3), \\ y = he_1 + j(de_2 - ce_3). \end{gathered}$$

Then,

$$\begin{gathered} a^2 + b^2 = 1, \\ c^2 + d^2 = 1, \\ f^2 + g^2 = 1, \\ h^2 + j^2 = 1. \end{gathered}$$

And, by 5,

$$\begin{gathered} R(v,w,w,v) = \epsilon, \\ R(v,u,u,v) = \delta, \\ R(w,x,x,w) = \delta, \\ R(u,y,y,u) = \epsilon. \end{gathered}$$

Hence,

$$\begin{aligned} R(v,w,w,v) &= R(e_1, ae_2 + be_3, ae_2 + be_3, e_1) \\ &= R(e_1, ae_2, ae_2, e_1) + R(e_1, ae_2, be_3, e_1) + R(e_1, be_3, ae_2, e_1) + \\ &\quad\; R(e_1, be_3, be_3, e_1) \\ &= a^2 R(e_1, e_2, e_2, e_1) + b^2 R(e_1, e_3, e_3, e_1) \\ &= \epsilon. \end{aligned}$$

$$\begin{aligned} R(v,u,u,v) &= R(e_1, ce_2 + de_3, ce_2 + de_3, e_1) \\ &= R(e_1, ce_2, ce_2, e_1) + R(e_1, ce_2, de_3, e_1) + R(e_1, de_3, ce_2, e_1) + \\ &\quad\; R(e_1, de_3, de_3, e_1) \\ &= c^2 R(e_1, e_2, e_2, e_1) + d^2 R(e_1, e_3, e_3, e_1) \\ &= \delta. \end{aligned}$$

$$
\begin{aligned}
R(w,x,x,w) &= R(ae_2+be_3, fe_1+g(be_2-ae_3), fe_1+g(be_2-ae_3), ae_2+be_3)\\
&= R(ae_2,fe_1,fe_1,ae_2)+R(ae_2,fe_1,fe_1,be_3)+R(ae_2,fe_1,gbe_2,ae_2)+\\
&\quad R(ae_2,fe_1,gbe_2,be_3)+R(ae_2,fe_1,-gae_3,ae_2)+R(ae_2,fe_1,-gae_3,be_3)+\\
&\quad R(ae_2,gbe_2,fe_1,ae_2)+R(ae_2,gbe_2,fe_1,be_3)+R(ae_2,gbe_2,gbe_2,ae_2)+\\
&\quad R(ae_2,gbe_2,gbe_2,be_3)+R(ae_2,gbe_2,-gae_3,ae_2)+R(ae_2,gbe_2,-gae_3,be_3)+\\
&\quad R(ae_2,-gae_3,fe_1,ae_2)+R(ae_2,-gae_3,fe_1,be_3)+R(ae_2,-gae_3,gbe_2,ae_2)+\\
&\quad R(ae_2,-gae_3,gbe_2,be_3)+R(ae_2,-gae_3,-gae_3,ae_2)+R(ae_2,-gae_3,-gae_3,be_3)+\\
&\quad R(be_3,fe_1,fe_1,ae_2)+R(be_3,fe_1,fe_1,be_3)+R(be_3,fe_1,gbe_2,ae_2)+\\
&\quad R(be_3,fe_1,gbe_2,be_3)+R(be_3,fe_1,-gae_3,ae_2)+R(be_3,fe_1,-gae_3,be_3)+\\
&\quad R(be_3,gbe_2,fe_1,ae_2)+R(be_3,gbe_2,fe_1,be_3)+R(be_3,gbe_2,gbe_2,ae_2)+\\
&\quad R(be_3,gbe_2,gbe_2,be_3)+R(be_3,gbe_2,-gae_3,ae_2)+R(be_3,gbe_2,-gae_3,be_3)+\\
&\quad R(be_3,-gae_3,fe_1,ae_2)+R(be_3,-gae_3,fe_1,be_3)+R(be_3,-gae_3,gbe_2,ae_2)+\\
&\quad R(be_3,-gae_3,gbe_2,be_3)+R(be_3,-gae_3,-gae_3,ae_2)+R(be_3,-gae_3,-gae_3,be_3)\\
&= f^2(a^2R(e_1,e_2,e_2,e_1)+b^2R(e_1,e_3,e_3,e_1))+g^2(a^2+b^2)^2R(e_2,e_3,e_3,e_2)\\
&= f^2(a^2R(e_1,e_2,e_2,e_1)+b^2R(e_1,e_3,e_3,e_1))+g^2R(e_2,e_3,e_3,e_2)\\
&= \delta.
\end{aligned}
$$

$$
\begin{aligned}
R(u,y,y,u) &= R(ce_2+de_3, he_1+j(de_2-ce_3), he_1+j(de_2-ce_3), ce_2+de_3)\\
&= R(ce_2,he_1,he_1,ce_2)+R(ce_2,he_1,he_1,de_3)+R(ce_2,he_1,jbe_2,ce_2)+\\
&\quad R(ce_2,he_1,jbe_2,de_3)+R(ce_2,he_1,-jce_3,ce_2)+R(ce_2,he_1,-jce_3,de_3)+\\
&\quad R(ce_2,jde_2,he_1,ce_2)+R(ce_2,jde_2,he_1,de_3)+R(ce_2,jde_2,jde_2,ce_2)+\\
&\quad R(ce_2,jde_2,jde_2,de_3)+R(ce_2,jde_2,-jce_3,ce_2)+R(ce_2,jde_2,-jce_3,de_3)+\\
&\quad R(ce_2,-jce_3,he_1,ce_2)+R(ce_2,-jce_3,he_1,de_3)+R(ce_2,-jce_3,jde_2,ce_2)+\\
&\quad R(ce_2,-jce_3,jde_2,de_3)+R(ce_2,-jce_3,-jce_3,ce_2)+R(ce_2,-jce_3,-jce_3,de_3)+\\
&\quad R(de_3,he_1,he_1,ce_2)+R(de_3,he_1,he_1,de_3)+R(de_3,he_1,jde_2,ce_2)+\\
&\quad R(de_3,he_1,jde_2,de_3)+R(de_3,he_1,-jce_3,ce_2)+R(de_3,he_1,-jce_3,de_3)+\\
&\quad R(de_3,jde_2,he_1,ce_2)+R(de_3,jde_2,he_1,de_3)+R(de_3,jde_2,jde_2,ce_2)+\\
&\quad R(de_3,jde_2,jde_2,de_3)+R(de_3,jde_2,-jce_3,ce_2)+R(de_3,jde_2,-jce_3,de_3)+\\
&\quad R(de_3,-jce_3,he_1,ce_2)+R(de_3,-jce_3,he_1,de_3)+R(de_3,-jce_3,jde_2,ce_2)+\\
&\quad R(de_3,-jce_3,jde_2,de_3)+R(de_3,-jce_3,-jce_3,ce_2)+R(de_3,-jce_3,-jce_3,de_3)\\
&= h^2(c^2R(e_1,e_2,e_2,e_1)+d^2R(e_1,e_3,e_3,e_1))+j^2(c^2+d^2)^2R(e_2,e_3,e_3,e_2)\\
&= h^2(c^2R(e_1,e_2,e_2,e_1)+d^2R(e_1,e_3,e_3,e_1))+j^2R(e_2,e_3,e_3,e_2)\\
&= \epsilon.
\end{aligned}
$$

So,

$$
\begin{aligned}
\epsilon &= h^2\delta + j^2R(e_2,e_3,e_3,e_2),\\
\delta &= f^2\epsilon + g^2R(e_2,e_3,e_3,e_2).
\end{aligned}
$$

So, multiplying the first equation by g^2 and the second equation by j^2, we see that

$$
\begin{aligned}
g^2\epsilon &= g^2h^2\delta + g^2j^2R(e_2,e_3,e_3,e_2),\\
j^2\delta &= j^2f^2\epsilon + g^2j^2R(e_2,e_3,e_3,e_2).
\end{aligned}
$$

So, subtracting the second equation from the first equation, we see that

$$g^2\epsilon - j^2\delta = g^2h^2\delta - j^2f^2\epsilon.$$

So, grouping similar terms together, we see that

$$g^2\epsilon + j^2 f^2\epsilon = j^2\delta + g^2h^2\delta.$$

So,

$$\begin{aligned}
g^2\epsilon + j^2 f^2\epsilon &= j^2\delta + g^2h^2\delta, \text{ which implies} \\
(g^2 + j^2 f^2)\epsilon &= (j^2 + g^2h^2)\delta, \text{ which implies} \\
(g^2 + j^2(1-g^2))\epsilon &= (j^2 + g^2(1-j^2))\delta, \text{ which implies} \\
(g^2 + j^2 - g^2j^2)\epsilon &= (g^2 + j^2 - g^2j^2)\delta.
\end{aligned}$$

Thus, either $\epsilon = \delta$ or $g^2 + j^2 - g^2j^2 = 0$. If $g^2 + j^2 - g^2j^2 = 0$, then since $g^2 + j^2 - g^2j^2 = g^2 + f^2j^2$, $g^2 = 0$ and $f^2j^2 = 0$. So, $g^2 = 0$ and either $f^2 = 0$ or $j^2 = 0$. Since $g^2 = 0$, $f^2 = 1 - g^2 = 1$, so $j^2 = 0$. Hence, $\delta = f^2\epsilon + g^2 R(e_2, e_3, e_3, e_2) = \epsilon$. Therefore, in either case, $\epsilon = \delta$. □

Theorem 2. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. $\mathcal{M}$ has $csc(\epsilon)$ if and only if*

$$\lambda_1 = \lambda_2 = \lambda_3 = 2\epsilon.$$

Proof. Assume $\mathcal{M}$ has $csc(\epsilon)$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$. Then,

$$R(e_1, e_2, e_2, e_1) = R(e_1, e_3, e_3, e_1) = R(e_2, e_3, e_3, e_2) = \epsilon.$$

So, by 14, $\lambda_1 = \lambda_2 = \lambda_3$. So,

$$\begin{aligned}
\tfrac{\lambda_i}{2} &= \tfrac{\lambda_i + \lambda_i - \lambda_i}{2} \\
&= \tfrac{\lambda_i + \lambda_j - \lambda_k}{2} \\
&= R(e_i, e_j, e_j, e_i) \\
&= \epsilon.
\end{aligned}$$

This implies $\lambda_i = 2\epsilon$. Hence,

$$\lambda_1 = \lambda_2 = \lambda_3 = 2\epsilon.$$

Assume

$$\lambda_1 = \lambda_2 = \lambda_3 = 2\epsilon.$$

Then,

$$R(e_1, e_2, e_2, e_3) = R(e_1, e_3, e_3, e_1) = R(e_2, e_3, e_3, e_2) = \epsilon.$$

Let v be in V, $v \neq 0$, let w be in V, $w \neq 0$, and let a, b, c, d, f, and g be constants such that

$$\begin{aligned}
v &= ae_1 + be_2 + ce_3, \\
w &= de_1 + fe_2 + ge_3.
\end{aligned}$$

Without loss of generality, assume $\langle v, v\rangle = 1$, $\langle w, w\rangle = 1$, and $\langle v, w\rangle = 0$.
Since $ad + bf + cg = 0$, we see that

$$\begin{aligned}
-2abdf - 2acdg - 2bcfg- & \\
2abdf - 2acdg - 2bcfg- & \\
2abdf - 2acdg - 2bcfg &= 2(-ad(bf + cg)) - 2bcfg- \\
&\quad 2abdf + 2(-cg(ad + bf))+ \\
&\quad 2(-bf(ad + cg)) - 2acdg \\
&= 2(-ad(-ad)) + 2(-cg(-cg)) + 2(-bf(-bf))- \\
&\quad 2bcfg - 2abdf - 2acdg, \text{ which implies} \\
2(-2abdf - 2acdg - 2bcfg) &= 2(a^2d^2 + c^2g^2 + b^2f^2), \text{ which implies} \\
-2abdf - 2acdg - 2bcfg &= a^2d^2 + c^2g^2 + b^2f^2.
\end{aligned}$$

In the following calculations, we omit R terms which are 0, in order to simplify the calculation

$$\begin{aligned}
R(v, w, w, v) &= R(ae_1 + be_2 + ce_3, de_1 + fe_2 + ge_3, de_1 + fe_2 + ge_3, ae_1 + be_2 + ce_3) \\
&= R(ae_1, fe_2, de_1, be_2) + R(ae_1, fe_2, fe_2, ae_1) + R(ae_1, ge_3, de_1, ce_3)+ \\
&\quad R(ae_1, ge_3, ge_3, ae_1) + R(be_2, de_1, de_1, be_2) + R(be_2, de_1, fe_2, ae_1)+ \\
&\quad R(be_2, ge_3, fe_2, ce_3) + R(be_2, ge_3, ge_3, be_2) + R(ce_3, de_1, de_1, ce_3)+ \\
&\quad R(ce_3, de_1, ge_3, ae_1) + R(ce_3, fe_2, fe_2, ce_3) + R(ce_3, fe_2, ge_3, be_2) \\
&= a^2f^2R(e_1, e_2, e_2, e_1) - 2abdfR(e_1, e_2, e_2, e_1) + b^2d^2R(e_1, e_2, e_2, e_1)+ \\
&\quad a^2g^2R(e_1, e_3, e_3, e_1) - 2acdgR(e_1, e_3, e_3, e_1) + c^2d^2R(e_1, e_3, e_3, e_1)+ \\
&\quad b^2g^2R(e_2, e_3, e_3, e_2) - 2bcfgR(e_2, e_3, e_3, e_2) + c^2f^2R(e_2, e_3, e_3, e_2) \\
&= (a^2f^2 - 2abdf + b^2d^2 + a^2g^2 - 2acdg + c^2d^2 + b^2g^2 - 2bcfg + c^2f^2)\epsilon \\
&= (a^2d^2 + a^2f^2 + a^2g^2 + b^2d^2 + b^2f^2 + b^2g^2 + c^2d^2 + c^2f^2 + c^2g^2)\epsilon \\
&= ((a^2 + b^2 + c^2)(d^2 + f^2 + g^2))\epsilon \\
&= (1)(1)\epsilon \\
&= \epsilon
\end{aligned}$$

Hence, since v and w were chosen arbitrarily, $\mathcal{M}$ has $csc(\epsilon)$. □

Theorem 3. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. $\mathcal{M}$ is Einstein, where c is some constant such that for all v in V and for all w in V, the Ricci tensor $\rho(v, w) = c\langle v, w\rangle$, if and only if M has $csc(\frac{c}{2})$.*

Proof. If:
Assume $\mathcal{M}$ has $csc(\frac{c}{2})$. Let A and $\{e_1, e_2, e_3\}$ be as in 10. Then, by 2,

$$\lambda_1 = \lambda_2 = \lambda_3 = c.$$

Suppose $R = R_\phi \pm R_\psi$. Then, without loss of generality, $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\lambda_1 + \lambda_2 = \lambda_3$. So, $c + c = c$, which implies that $c = \lambda_1 = 0$, a contradiction to the result that $\lambda_1 \neq 0$. Hence, $R = R_\phi$. Since $\mathcal{M}$ has $csc(\frac{c}{2})$, by 11, $R = \frac{c}{2}R_{\langle\cdot,\cdot\rangle}$. And,

$$R(e_1, e_2, e_2, e_1) = R(e_1, e_3, e_3, e_1) = R(e_2, e_3, e_3, e_2) = \frac{c}{2}.$$

Let v be in V, $v \neq 0$, let w be in V, $w \neq 0$, and let a, b, h, d, f, and g be constants such that

$$\begin{aligned} v &= ae_1 + be_2 + he_3, \\ w &= de_1 + fe_2 + ge_3. \end{aligned}$$

Then,

$$\begin{aligned} \rho(v,w) &= \rho(ae_1 + be_2 + he_3, de_1 + fe_2 + ge_3) \\ &= R(ae_1 + be_2 + he_3, e_1, e_1, de_1 + fe_2 + ge_3) + \\ &\quad R(ae_1 + be_2 + he_3, e_2, e_2, de_1 + fe_2 + ge_3) + \\ &\quad R(ae_1 + be_2 + he_3, e_3, e_3, de_1 + fe_2 + ge_3) \\ &= R(e_1, be_2, fe_2, e_1) + R(e_1, he_3, ge_3, e_1) + R(e_2, ae_1, de_1, e_2) + \\ &\quad R(e_2, he_3, ge_3, e_2) + R(e_3, ae_1, de_1, e_3) + R(e_3, be_2, fe_2, e_3) \\ &= bfR(e_2, e_1, e_1, e_2) + hgR(e_3, e_1, e_1, e_3) + adR(e_1, e_2, e_2, e_1) + \\ &\quad hgR(e_3, e_2, e_2, e_3) + adR(e_1, e_3, e_3, e_1) + bfR(e_2, e_3, e_3, e_2) \\ &= ad\rho(e_1, e_1) + bf\rho(e_2, e_2) + hg\rho(e_3, e_3) \\ &= ad\lambda_1 + bf\lambda_2 + hg\lambda_3 \\ &= (ad + bf + hg)c \\ &= c(\langle ae_1, de_1 \rangle + \langle be_2, fe_2 \rangle + \langle he_3, ge_3 \rangle) \\ &= c\langle ae_1 + be_2 + he_3, de_1 + fe_2 + ge_3 \rangle \\ &= c\langle v, w \rangle. \end{aligned}$$

Since v and w were chosen arbitrarily, $\rho = c\langle \cdot, \cdot \rangle$. Hence, $\mathcal{M}$ is Einstein.
Only if:
Assume $\mathcal{M}$ is Einstein. Let A and $\{e_1, e_2, e_3\}$ be as in 10. Since $\mathcal{M}$ is Einstein, for all v in V and for all w in V, $\rho(v,w) = c\langle v, w \rangle$ for some constant c.
Since $\rho(e_i, e_i) = \langle e_i, e_i \rangle \lambda_i = \lambda_i$, we see that

$$\begin{aligned} \lambda_i &= \rho(e_i, e_i) \\ &= c\langle e_i, e_i \rangle \\ &= c. \end{aligned}$$

Hence, $\lambda_1 = \lambda_2 = \lambda_3 = c$. So,

$$\begin{aligned} R(e_i, e_j, e_j, e_i) &= \tfrac{\lambda_i + \lambda_j - \lambda_k}{2} \\ &= \tfrac{\lambda_i + \lambda_i - \lambda_i}{2} \\ &= \tfrac{\lambda_i}{2} \\ &= \tfrac{c}{2}. \end{aligned}$$

So, $R = \frac{c}{2} R_{\langle \cdot, \cdot \rangle}$. Therefore, by 11, $\mathcal{M}$ has $csc(\frac{c}{2})$.

□

The following theorem is a strengthening of Lemma 2.3 in 6.

Theorem 4. *Let $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, where $dim(V) = 3$ and $\langle \cdot, \cdot \rangle$ is positive definite. Let $\{e_1, e_2, e_3\}$ be as in 2. Assume without loss of generality that $R(e_1, e_2, e_2, e_1) \geq R(e_1, e_3, e_3, e_1) \geq R(e_2, e_3, e_3, e_2)$. Assume $\mathcal{M}$ has $cvc(\epsilon)$. Then, $R(e_1, e_3, e_3, e_1) = \epsilon$.*

Proof. Let

$$\begin{aligned} R(e_1, e_2, e_2, e_1) &= \delta, \\ R(e_1, e_3, e_3, e_1) &= \eta, \text{ and} \\ R(e_2, e_3, e_3, e_2) &= \tau. \end{aligned}$$

Let $w_\theta = \cos\theta e_1 + \sin\theta e_2$. By 15, $f(\theta) =$

$$\begin{aligned} \kappa(span\{e_3, w_\theta\}) &= \cos^2\theta R(e_3, e_1, e_1, e_3) + \sin^2\theta R(e_3, e_2, e_3, e_3) + 2\cos\theta\sin\theta R(e_3, e_1, e_2, e_3) \\ &= \cos^2\theta\eta + \sin^2\theta\tau. \end{aligned}$$

Hence,

$$\begin{aligned} \tfrac{df}{d\theta} &= 2\cos\theta(-\sin\theta)\eta + 2\sin\theta\cos\theta\tau \\ &= 2\cos\theta\sin\theta(\tau - \eta). \\ \tfrac{d^2f}{d\theta^2} &= 2(\tau-\eta)(\tfrac{df}{d\theta}(\cos\theta)\sin\theta + \cos\theta\tfrac{df}{d\theta}(\sin\theta)) \\ &= 2(\tau-\eta)(-\sin\theta\sin\theta + \cos\theta\cos\theta) \\ &= 2(\tau-\eta)(\cos^2\theta - \sin^2\theta). \end{aligned}$$

Since $\tau \neq \eta$, for values of θ from 0 to π, $\frac{df}{d\theta} = 0$ when $\theta = 0$, $\theta = \frac{\pi}{2}$, or $\theta = \pi$.
Since $\tau < \eta$,

$$\begin{aligned} \kappa(span\{e_3, w_0\}) &= \cos^2 0\eta + \sin^2 0\tau \\ &= (1)\eta + (0)\tau \\ &= \eta. \\ \tfrac{d^2f}{d0^2} &= 2(\tau-\eta)(\cos^2 0 - \sin^2 0) \\ &= 2(\tau-\eta)((1)-(0)) \\ &= 2(\tau-\eta) < 0, \text{ which implies } \eta \text{ is a local maximum.} \\ \kappa(span\{e_3, w_{frac\pi 2}\}) &= \cos^2\tfrac{\pi}{2}\eta + \sin^2\tfrac{\pi}{2}\tau \\ &= (0)\eta + (1)\tau \\ &= \tau. \\ \tfrac{d^2f}{d\frac{\pi}{2}^2} &= 2(\tau-\eta)(\cos^2\tfrac{\pi}{2} - \sin^2\tfrac{\pi}{2}) \\ &= 2(\tau-\eta)((0)-(1)) \\ &= 2(\eta-\tau) > 0, \text{ which implies } \tau \text{ is a local minimum.} \\ \kappa(span\{e_3, w_\pi\}) &= \cos^2\pi\eta + \sin^2\pi\tau \\ &= (1)\eta + (0)\tau \\ &= \eta. \\ \tfrac{d^2f}{d\pi^2} &= 2(\tau-\eta)(\cos^2\pi - \sin^2\pi) \\ &= 2(\tau-\eta)((1)-(0)) \\ &= 2(\tau-\eta) < 0, \text{ which implies } \eta \text{ is a local maximum.} \end{aligned}$$

Since these are the critical points and endpoints of $f(\theta)$ for values of θ from 0 to π, and because $f(\theta)$ for values of θ from 0 to π represents every 2-plane which contains e_3, e_3 has sectional curvature between τ and η. Since $\mathcal{M}$ has $cvc(\epsilon)$, some nondegenerate 2-plane containing e_3 must have sectional curvature ϵ. Hence, $\tau \leq \epsilon \leq \eta$.
Let $u_\theta = \cos\theta e_2 + \sin\theta e_3$. By 15, $g(\theta) =$

$$\begin{aligned} \kappa(span\{e_1, u_\theta\}) &= \cos^2\theta R(e_1, e_2, e_2, e_1) + sin^2\theta R(e_1, e_3, e_3, e_1) + 2\cos\theta\sin\theta R(e_1, e_2, e_3, e_1) \\ &= \cos^2\theta\delta + \sin^2\theta\eta. \end{aligned}$$

Hence,

$$\begin{aligned}
\frac{dg}{d\theta} &= 2\cos\theta(-\sin\theta)\delta + 2\sin\theta\cos\theta\eta \\
&= 2\cos\theta\sin\theta(\eta-\delta). \\
\frac{d^2g}{d\theta^2} &= 2(\eta-\delta)(\tfrac{df}{d\theta}(\cos\theta)\sin\theta + \cos\theta\tfrac{df}{d\theta}(\sin\theta)) \\
&= 2(\eta-\delta)(-\sin\theta\sin\theta + \cos\theta\cos\theta) \\
&= 2(\eta-\delta)(\cos^2\theta - \sin^2\theta).
\end{aligned}$$

Since $\eta \neq \delta$, for values of θ from 0 to π, $\frac{df}{d\theta} = 0$ when $\theta = 0$, $\theta = \frac{\pi}{2}$, or $\theta = \pi$. Since $\eta < \delta$,

$$\begin{aligned}
\kappa(span\{e_3, w_0\}) &= \cos^2 0\delta + \sin^2 0\eta \\
&= (1)\delta + (0)\eta \\
&= \delta. \\
\frac{d^2g}{d0^2} &= 2(\eta-\delta)(\cos^2 0 - \sin^2 0) \\
&= 2(\eta-\delta)((1)-(0)) \\
&= 2(\eta-\delta) < 0, \text{ which implies } \delta \text{ is a local maximum.} \\
\kappa(span\{e_3, w_{frac\pi 2}\}) &= \cos^2\tfrac{\pi}{2}\delta + \sin^2\tfrac{\pi}{2}\eta \\
&= (0)\delta + (1)\eta \\
&= \eta. \\
\frac{d^2g}{d\frac{\pi}{2}^2} &= 2(\eta-\delta)(\cos^2\tfrac{\pi}{2} - \sin^2\tfrac{\pi}{2}) \\
&= 2(\eta-\delta)((0)-(1)) \\
&= 2(\delta-\eta) > 0, \text{ which implies } \eta \text{ is a local minimum.} \\
\kappa(span\{e_3, w_\pi\}) &= \cos^2\pi\delta + \sin^2\pi\eta \\
&= (1)\delta + (0)\eta \\
&= \delta. \\
\frac{d^2g}{d\pi^2} &= 2(\eta-\delta)(\cos^2\pi - \sin^2\pi) \\
&= 2(\eta-\delta)((1)-(0)) \\
&= 2(\eta-\delta) < 0, \text{ which implies } \delta \text{ is a local maximum.}
\end{aligned}$$

Since these are the critical points and endpoints of $f(\theta)$ for values of θ from 0 to π, and because $f(\theta)$ for values of θ from 0 to π represents every 2-plane which contains e_3, e_3 has sectional curvature between η and δ. Since $\mathcal{M}$ has $cvc(\epsilon)$, some nondegenerate 2-plane containing e_3 must have sectional curvature ϵ. Hence, $\eta \leq \epsilon \leq \delta$.
So, $\eta \leq \epsilon \leq \eta$. Therefore, $R(e_1, e_3, e_3, e_1) = \eta = \epsilon$. □

Theorem 5. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. $\mathcal{M}$ has $ecvc(\epsilon)$ if and only if $||spec(\rho)|| \leq 2$.*

Proof. We prove the logically equivalent statement that $\mathcal{M}$ does not have $ecvc(\epsilon)$ if and only if $||spec(\rho)|| = 3$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i, e_j) = \langle e_i, e_j\rangle\lambda_i$.
If:
Assume $||spec(\rho)|| = 3$. Then, $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, and $\lambda_2 \neq \lambda_3$. So, by 14, $R(e_1, e_2, e_2, e_1) \neq R(e_1, e_3, e_3, e_1)$, $R(e_1, e_2, e_2, e_1) \neq R(e_2, e_3, e_3, e_2)$, and $R(e_1, e_3, e_3, e_1) \neq R(e_2, e_3, e_3, e_2)$. Assume without loss of generality that $R(e_1, e_2, e_2, e_1) >$

$R(e_1,e_3,e_3,e_1) > R(e_2,e_3,e_3,e_2)$. Suppose $\mathcal{M}$ has $ecvc(\epsilon)$. Then, since $\mathcal{M}$ has $cvc(\epsilon)$, $R(e_1,e_3,e_3,e_1) = \epsilon$. But, then $R(e_1,e_2,e_2,e_1) > \epsilon > R(e_2,e_3,e_3,e_2)$. So, by 5, ϵ is neither an upper bound nor a lower bound on the sectional curvatures of the nondegenerate 2-planes, contradicting the assumption that $\mathcal{M}$ has $ecvc(\epsilon)$. So, $\mathcal{M}$ does not have $ecvc(\epsilon)$.
Only if:
Assume $||spec(\rho)|| \neq 3$. Then, $||spec(\rho)|| \leq 2$. If $||spec(\rho)|| = 1$, then by 2, $\mathcal{M}$ has $csc(\epsilon)$, which implies $\mathcal{M}$ has $ecvc(\epsilon)$. So, suppose $||spec(\rho)|| = 2$. Then, by 16, $\mathcal{M}$ has $cvc(\epsilon)$. Assume without loss of generality that $R(e_1,e_2,e_2,e_1) > R(e_1,e_3,e_3,e_1) > R(e_2,e_3,e_3,e_2)$. Then, by 4, $R(e_1,e_3,e_3,e_1) = \epsilon$. By 14, either $R(e_1,e_2,e_2,e_1) = R(e_1,e_3,e_3,e_1)$ or $R(e_2,e_3,e_3,e_2) = R(e_1,e_3,e_3,e_1)$.
Suppose $R(e_1,e_2,e_2,e_1) = R(e_1,e_3,e_3,e_1)$. Then, $R(e_1,e_2,e_2,e_1) = R(e_1,e_3,e_3,e_1) = \epsilon > R(e_2,e_3,e_3,e_2) = \delta = \epsilon - \tau$, where $\tau > 0$. Let v be in V, $v \neq 0$, w be in V, $w \neq 0$, and let a, b, c, d, f, and g be constants such that

$$\begin{aligned} v &= ae_1 + be_2 + ce_3, \\ w &= de_1 + fe_2 + ge_3. \end{aligned}$$

Without loss of generality, assume $\langle v,v\rangle = 1$, $\langle w,w\rangle = 1$, and $\langle v,w\rangle = 0$. Then,

$$\begin{aligned}
\kappa(span\{v,w\}) &= R(v,w,w,v) \\
&= R(ae_1+be_2+ce_3, de_1+fe_2+ge_3, de_1+fe_2+ge_3, ae_1+be_2+ce_3) \\
&= R(ae_1,fe_2,de_1,be_2) + R(ae_1,fe_2,fe_2,ae_1) + R(ae_1,ge_3,de_1,ce_3) + \\
&\quad R(ae_1,ge_3,ge_3,ae_1) + R(be_2,de_1,de_1,be_2) + R(be_2,de_1,fe_2,ae_1) + \\
&\quad R(be_2,ge_3,fe_2,ce_3) + R(be_2,ge_3,ge_3,be_2) + R(ce_3,de_1,de_1,ce_3) + \\
&\quad R(ce_3,de_1,ge_3,ae_1) + R(ce_3,fe_2,fe_2,ce_3) + R(ce_3,fe_2,ge_3,be_2) \\
&= a^2f^2R(e_1,e_2,e_2,e_1) - 2abdfR(e_1,e_2,e_2,e_1) + b^2d^2R(e_1,e_2,e_2,e_1) + \\
&\quad a^2g^2R(e_1,e_3,e_3,e_1) - 2acdgR(e_1,e_3,e_3,e_1) + c^2d^2R(e_1,e_3,e_3,e_1) + \\
&\quad b^2g^2R(e_2,e_3,e_3,e_2) - 2bcfgR(e_2,e_3,e_3,e_2) + c^2f^2R(e_2,e_3,e_3,e_2) \\
&= a^2f^2\epsilon - 2abdf\epsilon b^2d^2\epsilon + a^2g^2\epsilon - 2acdg\epsilon + c^2d^2\epsilon + \\
&\quad b^2g^2\delta - 2bcfg\delta + c^2f^2\delta \\
&= a^2f^2\epsilon - 2abdf\epsilon b^2d^2\epsilon + a^2g^2\epsilon - 2acdg\epsilon + c^2d^2\epsilon + \\
&\quad b^2g^2(\epsilon-\tau) - 2bcfg(\epsilon-\tau) + c^2f^2(\epsilon-\tau) \\
&= (a^2f^2 - 2abdf + b^2d^2 + a^2g^2 - 2acdg + c^2d^2 + b^2g^2 - 2bcfg + c^2f^2)\epsilon + \\
&\quad (-b^2g^2 + 2bcfg - c^2f^2)\tau \\
&= (a^2d^2 + a^2f^2 + a^2g^2 + b^2d^2 + b^2f^2 + b^2g^2 + c^2d^2 + c^2f^2 + c^2g^2)\epsilon + \\
&\quad (-b^2g^2 + 2bcfg - c^2f^2)\tau \\
&= ((a^2+b^2+c^2)(d^2+f^2+g^2))\epsilon + (-b^2g^2 + 2bcfg - c^2f^2)\tau \\
&= (1)(1)\epsilon + (-b^2g^2 + 2bcfg - c^2f^2)\tau \\
&= \epsilon - (bg-cf)^2\tau \leq \epsilon.
\end{aligned}$$

Hence, since v and w were chosen arbitrarily, for all v in V and for all w in V, $\kappa(span\{v,w\}) \leq \epsilon$.
Suppose $R(e_2,e_3,e_3,e_2) = R(e_1,e_3,e_3,e_1)$. Then, $\epsilon + \tau = \delta = R(e_1,e_2,e_2,e_1) > R(e_1,e_3,e_3,e_1)\epsilon = R(e_2,e_3,e_3,e_2)$, where $\tau > 0$. Let v be in V, $v \neq 0$, w be in V, $w \neq 0$, and let a, b, c, d, f, and g be constants such that

$$\begin{aligned} v &= ae_1 + be_2 + ce_3, \\ w &= de_1 + fe_2 + ge_3. \end{aligned}$$

Without loss of generality, assume $\langle v,v\rangle = 1$, $\langle w,w\rangle = 1$, and $\langle v,w\rangle = 0$. Then,

$$\begin{aligned}
\kappa(span\{v,w\}) &= R(v,w,w,v)\\
&= R(ae_1+be_2+ce_3, de_1+fe_2+ge_3, de_1+fe_2+ge_3, ae_1+be_2+ce_3)\\
&= R(ae_1,fe_2,de_1,be_2)+R(ae_1,fe_2,fe_2,ae_1)+R(ae_1,ge_3,de_1,ce_3)+\\
&\quad R(ae_1,ge_3,ge_3,ae_1)+R(be_2,de_1,de_1,be_2)+R(be_2,de_1,fe_2,ae_1)+\\
&\quad R(be_2,ge_3,fe_2,ce_3)+R(be_2,ge_3,ge_3,be_2)+R(ce_3,de_1,de_1,ce_3)+\\
&\quad R(ce_3,de_1,ge_3,ae_1)+R(ce_3,fe_2,fe_2,ce_3)+R(ce_3,fe_2,ge_3,be_2)\\
&= a^2f^2R(e_1,e_2,e_2,e_1)-2abdfR(e_1,e_2,e_2,e_1)+b^2d^2R(e_1,e_2,e_2,e_1)+\\
&\quad a^2g^2R(e_1,e_3,e_3,e_1)-2acdgR(e_1,e_3,e_3,e_1)+c^2d^2R(e_1,e_3,e_3,e_1)+\\
&\quad b^2g^2R(e_2,e_3,e_3,e_2)-2bcfgR(e_2,e_3,e_3,e_2)+c^2f^2R(e_2,e_3,e_3,e_2)\\
&= a^2f^2\epsilon-2abdf\epsilon b^2d^2\epsilon+a^2g^2\epsilon-2acdg\epsilon+c^2d^2\epsilon+\\
&\quad b^2g^2\delta-2bcfg\delta+c^2f^2\delta\\
&= a^2f^2\epsilon-2abdf\epsilon b^2d^2\epsilon+a^2g^2\epsilon-2acdg\epsilon+c^2d^2\epsilon+\\
&\quad b^2g^2(\epsilon+\tau)-2bcfg(\epsilon+\tau)+c^2f^2(\epsilon+\tau)\\
&= (a^2f^2-2abdf+b^2d^2+a^2g^2-2acdg+c^2d^2+b^2g^2-2bcfg+c^2f^2)\epsilon+\\
&\quad (b^2g^2-2bcfg+c^2f^2)\tau\\
&= (a^2d^2+a^2f^2+a^2g^2+b^2d^2+b^2f^2+b^2g^2+c^2d^2+c^2f^2+c^2g^2)\epsilon+\\
&\quad (b^2g^2-2bcfg+c^2f^2)\tau\\
&= ((a^2+b^2+c^2)(d^2+f^2+g^2))\epsilon+(b^2g^2-2bcfg+c^2f^2)\tau\\
&= (1)(1)\epsilon+(b^2g^2-2bcfg+c^2f^2)\tau\\
&= \epsilon+(bg-cf)^2\tau \geq \epsilon.
\end{aligned}$$

Hence, since v and w were chosen arbitrarily, for all v in V and for all w in V, $\kappa(span\{v,w\}) \geq \epsilon$. Therefore, $\mathcal{M}$ has $ecvc(\epsilon)$. □

Corollary 3. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. If $\lambda_i = \lambda_j$, then $\mathcal{M}$ has $ecvc(\frac{\lambda_k}{2})$.*

Proof. Assume without loss of generality that $\lambda_i = \lambda_j$. By 16, $\mathcal{M}$ has $cvc(\frac{\lambda_k}{2})$. By 5, $\mathcal{M}$ has $ecvc(\epsilon)$. Therefore, by 1, $\mathcal{M}$ has $ecvc(\frac{\lambda_k}{2})$. □

Theorem 6. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$ and $\langle\cdot,\cdot\rangle$ is positive definite. If $\mathcal{M}$ has $cvc(0)$ and $||spec(\rho)|| = 3$, then $R = R_\phi \pm R_\psi$.*

Proof. Assume $\mathcal{M}$ has $cvc(0)$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ such that the Ricci tensor $\rho(e_i,e_j) = \langle e_i,e_j\rangle\lambda_i$. Assume $||spec(\rho)|| = 3$. Then, by 14, $R(e_1,e_2,e_2,e_1) \neq R(e_1,e_3,e_3,e_1)$, $R(e_1,e_2,e_2,e_1) \neq R(e_2,e_3,e_3,e_2)$, and $R(e_1,e_3,e_3,e_1) \neq R(e_2,e_3,e_3,e_2)$. Assume without loss of generality that $R(e_1,e_2,e_2,e_1) > R(e_1,e_3,e_3,e_1) > R(e_2,e_3,e_3,e_2)$. Then, by 4, $R(e_1,e_3,e_3,e_1) = 0$. So,

$$\begin{aligned}
\tfrac{\lambda_1+\lambda_3-\lambda_2}{2} &= R(e_1,e_3,e_3,e_1)\\
&= 0, \text{ which implies}\\
\lambda_1+\lambda_3 &= \lambda_2.
\end{aligned}$$

But,

$$\begin{aligned} R(e_1,e_2,e_2,e_1) &= \tfrac{\lambda_1+\lambda_2-\lambda_3}{2} \\ &= \tfrac{\lambda_1+\lambda_1+\lambda_3-\lambda_3}{2} \\ &= \tfrac{2\lambda_1}{2} \\ &= \lambda_1 \neq 0. \end{aligned}$$

By analogous reasoning, $R(e_2,e_3,e_3,e_2) = \lambda_3 \neq 0$. So, $\lambda_1 \neq 0$, $\lambda_3 \neq 0$, and $\lambda_1 + \lambda_3 = \lambda_2$. Therefore, by 12, $R = R_\phi \pm R_\psi$. □

Theorem 7. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$, $\langle\cdot,\cdot\rangle$ is positive definite, and $R = R_\phi$. Let $\{e_1,e_2,e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ. If $\mathcal{M}$ has $cvc(\epsilon)$ and*

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_1), R(e_2,e_3,e_3,e_2)\}|| = 3,$$

then $\epsilon > 0$.

Proof. Without loss of generality, assume $R(e_1,e_2,e_2,e_1) > R(e_1,e_3,e_3,e_1) > R(e_2,e_3,e_3,e_2)$. By 4, $R(e_1,e_3,e_3,e_1) = \epsilon$. By 6, assume without loss of generality that either $\epsilon = -1$, $\epsilon = 0$, or $\epsilon = 1$. By the contrapositive of 6, $\epsilon \neq 0$. Suppose $\epsilon = -1$. So, $R(e_1,e_3,e_3,e_1) = \eta_1\eta_3 = -1$, which implies $\eta_3 = \frac{-1}{\eta_1}$. Since $\mathcal{M}$ has $cvc(-1)$, for all v in V, $v \neq 0$, there exists w in V such that $\kappa(span\{v,w\}) = -1$. Without loss of generality, assume $\langle v,v\rangle = 1$, $\langle w,w\rangle = 1$, $\langle v,w\rangle = 0$, and $\phi(v,w) = 0$.

$$\begin{aligned} \kappa(span\{v,w\}) &= \tfrac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle\langle w,v\rangle} \\ &= \tfrac{R_\phi(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle\langle w,v\rangle} \\ &= \tfrac{\phi(v,v)\phi(w,w)-\phi(v,w)\phi(w,v)}{\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle\langle w,v\rangle} \\ &= \phi(v,v)\phi(w,w), \text{ which implies} \\ \phi(v,v)\phi(w,w) &= -1. \end{aligned}$$

We show that there exists a v in V, $v \neq 0$, such that $\phi(v,v) = 0$, so that for all w in V, $\phi(v,v)\phi(w,w) = 0 \neq -1$.
If $\eta_1 = 0$, then $R(e_1,e_3,e_3,e_1) = \eta_1\eta_3 = 0$, contradicting the assumption that $R(e_1,e_3,e_3,e_1) = -1$. So, $\eta_1 \neq 0$. If $\eta_1 = \eta_2$, then

$$R(e_1,e_3,e_3,e_1) = \eta_1\eta_3 = \eta_2\eta_3 = R(e_2,e_3,e_3,e_2),$$

contradicting the assumption that

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_1), R(e_2,e_3,e_3,e_2)\}|| = 3.$$

So, $\eta_1 \neq \eta_2$. Let a, b, and c be constants such that $v = ae_1 + be_2 + ce_3$. Let

$$\begin{aligned} a &= \sqrt{\tfrac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)}}, \\ b &= \sqrt{\tfrac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)}}, \text{and} \\ |c| &\leq 1. \end{aligned}$$

So,

$$
\begin{aligned}
a^2+b^2+c^2 &= \sqrt{\frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)}}^2 + \sqrt{\frac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)}}^2 + c^2 \\
&= \frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)} + \frac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)} + c^2 \\
&= \frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)} + \frac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)} + \frac{c^2\eta_1(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2-c^2-c^2\eta_1\eta_1+\eta_1\eta_1+c^2\eta_1(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{c^2\eta_1(\eta_1-\eta_2)-c^2\eta_1(\eta_1-\eta_2)+\eta_1(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{\eta_1(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= 1.
\end{aligned}
$$

Hence, $\langle v, v\rangle = 1$, as required. But,

$$
\begin{aligned}
\phi(v,v) &= \phi(ae_1+be_2+ce_3, ae_1+be_2+ce_3) \\
&= a^2\phi(e_1,e_1,)+b^2\phi(e_2,e_2)+c^2\phi(e_3,e_3) \\
&= a^2\eta_1+b^2\eta_2+c^2\eta_3 \\
&= a^2\eta_1+b^2\eta_2-\frac{c^2}{\eta_1} \\
&= \sqrt{\frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)}}^2\eta_1 + \sqrt{\frac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)}}^2\eta_2 - \frac{c^2}{\eta_1} \\
&= \frac{c^2+c^2\eta_1\eta_2-\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)}\eta_1 + \frac{-c^2-c^2\eta_1\eta_1+\eta_1\eta_1}{\eta_1(\eta_1-\eta_2)}\eta_2 - \frac{c^2(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{c^2\eta_1+c^2\eta_1\eta_1\eta_2-\eta_1\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)} + \frac{-c^2\eta_2-c^2\eta_1\eta_1\eta_2+\eta_1\eta_1\eta_2}{\eta_1(\eta_1-\eta_2)} - \frac{c^2(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{c^2\eta_1+c^2\eta_1\eta_1\eta_2-\eta_1\eta_1\eta_2-c^2\eta_2-c^2\eta_1\eta_1\eta_2+\eta_1\eta_1\eta_2-c^2(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{c^2(\eta_1-\eta_2)-c^2(\eta_1-\eta_2)}{\eta_1(\eta_1-\eta_2)} \\
&= \frac{0}{\eta_1(\eta_1-\eta_2)} \\
&= 0.
\end{aligned}
$$

Hence, $\mathcal{M}$ does not have $cvc(-1)$, contradicting the assumption that $\mathcal{M}$ has $cvc(-1)$. So, $\epsilon \neq -1$. Hence, $\epsilon = 1$. Therefore, $\epsilon > 0$. □

Theorem 8. *Let $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$, $\langle\cdot,\cdot\rangle$ is positive definite, and $R = R_\phi$. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ. If $\mathcal{M}$ has $cvc(\epsilon)$ and*

$$||\{R(e_1,e_2,e_2,e_1), R(e_1,e_3,e_3,e_1), R(e_2,e_3,e_3,e_2)\}|| = 3,$$

then either

$$
\begin{aligned}
\eta_1 &= -\sqrt{\tfrac{\delta}{\tau}}, \\
\eta_2 &= -\sqrt{\delta\tau}, \text{ and} \\
\eta_3 &= -\sqrt{\tfrac{\tau}{\delta}},
\end{aligned}
$$

or

$$
\begin{aligned}
\eta_1 &= \sqrt{\tfrac{\delta}{\tau}}, \\
\eta_2 &= \sqrt{\delta\tau}, \text{ and} \\
\eta_3 &= \sqrt{\tfrac{\tau}{\delta}},
\end{aligned}
$$

where

$$R(e_1, e_2, e_2, e_1) = \delta > R(e_1, e_3, e_3, e_1) = \epsilon > R(e_2, e_3, e_3, e_2) = \tau.$$

Proof. By 7, $\epsilon > 0$. By 6, without loss of generality, assume $\epsilon = 1$. Then, $R(e_1, e_2, e_2, e_1) = \eta_1\eta_2 = \delta$, $R(e_1, e_3, e_3, e_1) = \eta_1\eta_3 = 1$, and $R(e_2, e_3, e_3, e_2) = \eta_2\eta_3 = \tau$. If $0 > \eta_3 > \eta_2 > \eta_1$, then

$$\begin{aligned}
\eta_1 &= -\sqrt{\eta_1^2} \\
&= -\sqrt{\tfrac{\eta_1}{\eta_3}} \\
&= -\sqrt{\tfrac{\eta_1\eta_2}{\eta_2\eta_3}} \\
&= -\sqrt{\tfrac{\delta}{\tau}}, \\
\eta_2 &= -\sqrt{\eta_2^2} \\
&= -\sqrt{\eta_1\eta_2\eta_2\eta_3} \\
&= -\sqrt{\delta\tau}, \text{ and} \\
\eta_3 &= -\tfrac{1}{\eta_1} \\
&= -\sqrt{\tfrac{1}{\eta_1\eta_1}} \\
&= -\sqrt{\tfrac{\eta_3}{\eta_1}} \\
&= -\sqrt{\tfrac{\eta_2\eta_3}{\eta_1\eta_2}} \\
&= -\sqrt{\tfrac{\tau}{\delta}}.
\end{aligned}$$

If $\eta_1 > \eta_2 > \eta_3 > 0$, then

$$\begin{aligned}
\eta_1 &= \sqrt{\eta_1^2} \\
&= \sqrt{\tfrac{\eta_1}{\eta_3}} \\
&= \sqrt{\tfrac{\eta_1\eta_2}{\eta_2\eta_3}} \\
&= \sqrt{\tfrac{\delta}{\tau}}, \\
\eta_2 &= \sqrt{\eta_2^2} \\
&= \sqrt{\eta_1\eta_2\eta_2\eta_3} \\
&= \sqrt{\delta\tau}, \text{ and} \\
\eta_3 &= \tfrac{1}{\eta_1} \\
&= \sqrt{\tfrac{1}{\eta_1\eta_1}} \\
&= \sqrt{\tfrac{\eta_3}{\eta_1}} \\
&= \sqrt{\tfrac{\eta_2\eta_3}{\eta_1\eta_2}} \\
&= \sqrt{\tfrac{\tau}{\delta}}.
\end{aligned}$$

□

Theorem 9. *There exists $\mathcal{M} = (V, \langle\cdot,\cdot\rangle, R)$, where $dim(V) = 3$, $\langle\cdot,\cdot\rangle$ is positive definite, and $R = R_\phi$ such that $||spec(\rho)|| = 3$ and $\mathcal{M}$ has $cvc(\epsilon)$.*

Proof. Let $\{e_1, e_2, e_3\}$ be a basis for V which is orthonormal with respect to $\langle\cdot,\cdot\rangle$ and orthogonal with respect to ϕ. By 7, if $\mathcal{M}$ has $cvc(\epsilon)$, then $\epsilon > 0$. Without loss of generality by 6, we show there exists $\mathcal{M}$ which has $cvc(1)$.
Let

$$\begin{aligned} R(e_1,e_2,e_2,e_1) &= \eta_1\eta_2 &=& \quad \delta > 1, \\ R(e_1,e_2,e_2,e_1) &= \eta_1\eta_3 &=& \quad 1, \text{ and} \\ R(e_2,e_3,e_3,e_2) &= \eta_2\eta_3 &=& \quad \tau < 1. \end{aligned}$$

By 19, $\eta_1 > \eta_2 > \eta_3 > 0$, so ϕ is positive definite. We show that for all v in V, $v \neq 0$, there exists w in V such that $\kappa(span\{v,w\}) = 1$. Without loss of generality, let $\langle v,v\rangle = 1$, $\langle w,w\rangle = 1$, $\langle v,w\rangle = 0$, and $\phi(v,w) = 0$. Let a, b, c, d, f, and g be constants such that

$$\begin{aligned} v &= ae_1 + be_2 + ce_3 \neq 0 \text{ and} \\ w &= de_1 + fe_2 + ge_3. \end{aligned}$$

Then,

$$\begin{aligned} \kappa(span\{v,w\}) &= \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\ &= \frac{R_\phi(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\ &= \frac{\phi(v,v)\phi(w,w) - \phi(v,w)\phi(w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\ &= \frac{\phi(v,v)\phi(w,w) - (0)(0)}{(1)(1)-(0)(0)} \\ &= \phi(v,v)\phi(w,w) \\ &= \phi(ae_1+be_2+ce_3, ae_1+be_2+ce_3)\phi(de_1+fe_2+ge_3, de_1+fe_2+ge_3) \\ &= (a^2\phi(e_1,e_1) + b^2\phi(e_2,e_2) + c^2\phi(e_3,e_3))(d^2\phi(e_1,e_1) + f^2\phi(e_2,e_2) + g^2\phi(e_3,e_3)) \\ &= (a^2\eta_1 + b^2\eta_2 + c^2\eta_3)(d^2\eta_1 + f^2\eta_2 + g^2\eta_3) \\ &= \phi(v,v)(d^2\eta_1 + f^2\eta_2 + g^2\eta_3). \end{aligned}$$

Since $\langle\cdot,\cdot\rangle$ and ϕ are both positive definite and $v \neq 0$, $\phi(v,v) \neq 0$, so we find w such that $(a^2\eta_1 + b^2\eta_2 + \frac{c^2}{\eta_1})(d^2\eta_1 + f^2\eta_2 + \frac{g^2}{\eta_1}) = 1$. To do this, we use the Lagrange Multipliers Method and the Intermediate Value Theorem. We first find the extrema of $h_1(a,b,c) = \phi(v,v) = a^2\eta_1 + b^2\eta_2 + c^2\eta_3$ subject to the condition that $j_1(a,b,c) = 1 - a^2 - b^2 - c^2 = 0$. The extrema occur when there exists a formula σ_1 such that $\nabla h_1(a,b,c) + \sigma_1\nabla j_1(a,b,c) = 0$. So, we find σ_1 such that

$$\begin{aligned} 2a\eta_1 - 2a\sigma_1 = 0, \\ 2b\eta_2 - 2b\sigma_1 = 0, \text{ and} \\ 2c\eta_3 - 2c\sigma_1 = 0. \end{aligned}$$

We see that

$$\begin{aligned} 0 &= 2a\eta_1 - 2a\sigma_1 \\ &= 2a(\eta_1 - \sigma_1) \\ &= a(\eta_1 - \sigma_1), \\ 0 &= 2b\eta_2 - 2b\sigma_1 \\ &= 2b(\eta_2 - \sigma_1), \\ &= b(\eta_2 - \sigma_1), \text{ and} \\ 0 &= 2c\eta_3 - 2c\sigma_1 \\ &= 2c(\eta_3 - \sigma_1) \\ &= c(\eta_3 - \sigma_1). \end{aligned}$$

So, $a = 0$ or $\sigma_1 = \eta_1$, $b = 0$ or $\sigma_1 = \eta_2$, and $c = 0$ or $\sigma_1 = \eta_3$. Since $v \neq 0$, either $a \neq 0$, $b \neq 0$, or $c \neq 0$. So, either $\sigma_1 = \eta_1$, $\sigma_1 = \eta_2$, or $\sigma_1 = \eta_3$. If $sigma_1 = \eta_3$, then $a = 0$ and $b = 0$, which implies $c = -1$ or $c = 1$. If $sigma_1 = \eta_2$, then $a = 0$ and $c = 0$, which implies $b = -1$ or $b = 1$. So, $v = -e_2$ or $v = e_2$. If $sigma_1 = \eta_1$, then $b = 0$ and $c = 0$, which implies $a = -1$ or $a = 1$. So, $v = -e_1$ or $v = e_1$. So, $v = -e_3$ or $v = e_3$. Hence, $v = -e_3$, $v = e_3$, $v = -e_2$, $v = e_2$, $v = -e_1$, or $v = e_1$. So, the extrema of $h_1(a, b, c) = \phi(v, v)$ are $\phi(-e_3, -e_3) = \phi(e_3, e_3) = \eta_3$, $\phi(-e_2, -e_2) = \phi(e_2, e_2) = \eta_2$, and $\phi(-e_1, -e_1) = \phi(e_1, e_1) = \eta_1$. Since $0 < \eta_3 < \eta_2 < \eta_1$, η_3 is the global minimum of $\phi(v, v)$ and η_1 is the global maximum of $\phi(v, v)$. So, $\eta_3 \leq \phi(v, v) \leq \eta_1$.

We find the extrema of $h_2(d, f, g) = \phi(v, v)\phi(w, w) = \phi(v, v)(d^2\eta_1 + f^2\eta_2 + g^2\eta_3)$ subject to the conditions that $j_2(d, f, g) = 1 - d^2 - f^2 - g^2 = 0$ and $j_3(d, f, g) = ad + bf + cg = 0$. The extrema occur when there exist formulae σ_2 and σ_3 such that $\nabla h_2(d, f, g) + \sigma_2 \nabla j_2(d, f, g) + \sigma_3 \nabla j_3(d, f, g) = 0$. So, we find σ_2 and σ_3 such that

$$\begin{aligned} 2d\eta_1\phi(v, v) - 2d\sigma_2 + a\sigma_3 &= 0, \\ 2f\eta_2\phi(v, v) - 2f\sigma_2 + b\sigma_3 &= 0, \text{and} \\ 2g\eta_3\phi(v, v) - 2g\sigma_2 + c\sigma_3 &= 0. \end{aligned}$$

From this set of equations, we see that by multiplying each equation by d, f, and g, respectively, it is the case that

$$\begin{aligned} 2d^2\eta_1\phi(v, v) - 2d^2\sigma_2 + ad\sigma_3 &= 0, \\ 2f^2\eta_2\phi(v, v) - 2f^2\sigma_2 + bf\sigma_3 &= 0, \text{ and} \\ 2g^2\eta_3\phi(v, v) - 2g^2\sigma_2 + cg\sigma_3 &= 0. \end{aligned}$$

The sum of the three equations is

$$\begin{aligned} 0 &= 0 + 0 + 0 \\ &= 2d^2\eta_1\phi(v, v) - 2d^2\sigma_2 + ad\sigma_3 + \\ &\quad 2f^2\eta_2\phi(v, v) - 2f^2\sigma_2 + bf\sigma_3 + \\ &\quad 2g^2\eta_3\phi(v, v) - 2g^2\sigma_2 + cg\sigma_3 \\ &= 2(d^2\eta_1 + f^2\eta_2 + g^2\eta_3)\phi(v, v) - 2(d^2 + f^2 + g^2)\sigma_2 + (ad + bf + cg)\sigma_3 \\ &= 2\phi(w, w)\phi(v, v) - 2(1)\sigma_2 + (0)\sigma_3 \\ &= 2\phi(v, v)\phi(w, w) - 2\sigma_2 \\ &= \phi(v, v)\phi(w, w) - \sigma_2, \text{ which implies} \\ \sigma_2 &= \phi(v, v)\phi(w, w). \end{aligned}$$

Also from the set of equations, we see that by multiplying each equation by a, b, and c, respectively, it is the case that

$$\begin{aligned} 2ad\eta_1\phi(v, v) - 2ad\sigma_2 + a^2\sigma_3 &= 0, \\ 2bf\eta_2\phi(v, v) - 2bf\sigma_2 + b^2\sigma_3 &= 0, \text{ and} \\ 2cg\eta_3\phi(v, v) - 2cg\sigma_2 + c^2\sigma_3 &= 0. \end{aligned}$$

The sum of these three equations is

$$\begin{aligned}
0 &= 0+0+0 \\
&= 2ad\eta_1\phi(v,v) - 2ad\sigma_2 + a^2\sigma_3 + \\
&\quad 2bf\eta_2\phi(v,v) - 2bf\sigma_2 + b^2\sigma_3 + \\
&\quad 2cg\eta_3\phi(v,v) - 2cg\sigma_2 + c^2\sigma_3 \\
&= 2(ad\eta_1 + bf\eta_2 + cg\eta_3)\phi(v,v) - 2(ad+bf+cg)\sigma_2 + (a^2+b^2+c^2)\sigma_3 \\
&= 2(ad\phi(e_1,e_1) + bf\phi(e_2,e_2) + cg\phi(e_3,e_3))\phi(v,v) - 2(0)\sigma_2 + (1)\sigma_3 \\
&= 2(\phi(ae_1,de_1) + \phi(be_2,fe_2) + \phi(ce_3,ge_3))\phi(v,v) + \sigma_3 \\
&= 2\phi(ae_1+be_2+ce_3, de_1+fe_2+ge_3)\phi(v,v) + \sigma_3 \\
&= 2\phi(v,w)\phi(v,v) + \sigma_3 \\
&= 2(0)\phi(v,v) + \sigma_3 \\
&= \sigma_3, \text{ which implies} \\
\sigma_3 &= 0.
\end{aligned}$$

So,

$$\begin{aligned}
0 &= 2d\eta_1\phi(v,v) - 2d\phi(v,v)\phi(w,w) + a(0) \\
&= 2d\phi(v,v)(\eta_1 - \phi(w,w)) \\
&= d\phi(v,v)(\eta_1 - \phi(w,w)), \\
0 &= 2f\eta_2\phi(v,v) - 2f\phi(v,v)\phi(w,w) + b(0) \\
&= 2f\phi(v,v)(\eta_2 - \phi(w,w)), \\
&= f\phi(v,v)(\eta_2 - \phi(w,w)), \\
0 &= 2g\eta_3\phi(v,v) - 2g\phi(v,v)\phi(w,w) + c(0) \\
&= 2g\phi(v,v)(\eta_3 - \phi(w,w)), \text{ and} \\
&= g\phi(v,v)(\eta_3 - \phi(w,w)).
\end{aligned}$$

Since $v \neq 0$ and ϕ is positive definite, $\phi(v,v) \neq 0$. This implies either $d = 0$ or $\phi(w,w) = \eta_1$, either $f = 0$ or $\phi(w,w) = \eta_2$, and either $g = 0$ or $\phi(w,w) = \eta_3$. Since $w \neq 0$, either $d \neq 0$, $f \neq 0$, or $g \neq 0$. So, either $\phi(w,w) = \eta_3$, $\phi(w,w) = \eta_2$, or $\phi(w,w) = \eta_1$. If $\phi(w,w) = \eta_3$, then since $\eta_3 \neq \eta_1$ and $\eta_3 \neq \eta_2$, $\phi(w,w) \neq \eta_1$ and $\phi(w,w) \neq \eta_2$, so $d = 0$ and $f = 0$, which implies $g = -1$ or $g = 1$. So, $w = -e_3$ or $w = e_3$. If $\phi(w,w) = \eta_2$, then since $\eta_2 \neq \eta_1$ and $\eta_2 \neq \eta_3$, $\phi(w,w) \neq \eta_1$ and $\phi(w,w) \neq \eta_3$, so $d = 0$ and $g = 0$, which implies $f = -1$ or $f = 1$. So, $w = -e_2$ or $w = e_2$. If $\phi(w,w) = \eta_1$, then since $\eta_1 \neq \eta_2$ and $\eta_1 \neq \eta_3$, $\phi(w,w) \neq \eta_2$ and $\phi(w,w) \neq \eta_3$, so $f = 0$ and $g = 0$, which implies $d = -1$ or $d = 1$. So, $w = -e_1$ or $w = e_1$. Hence, $w = -e_3$, $w = e_3$, $w = -e_2$, $w = e_2$, $w = -e_1$, or $w = e_1$. So, the extrema of $h_2(d,f,g) = \phi(v,v)\phi(w,w)$ are $\phi(v,v)\phi(-e_3,-e_3) = \phi(v,v)\phi(e_3,e_3) = \eta_3\phi(v,v)$, $\phi(v,v)\phi(-e_2,-e_2) = \phi(v,v)\phi(e_2,e_2) = \eta_2\phi(v,v)$, and $\phi(v,v)\phi(-e_1,-e_1) = \phi(v,v)\phi(e_1,e_1) = \eta_1\phi(v,v)$.
Since $v \neq 0$ and ϕ is positive definite, $\phi(v,v) > 0$. So, since $0 < \eta_3 < \eta_2 < \eta_1$, $\eta_3\phi(v,v) < \eta_2\phi(v,v) < \eta_1\phi(v,v)$. So, $\eta_3\phi(v,v)$ is the global minimum of $\phi(v,v)\phi(w,w)$ and $\eta_1\phi(v,v)$ is the global maximum of $\phi(v,v)\phi(w,w)$. Hence, $\eta_3\phi(v,v) \leq \phi(v,v)\phi(w,w) \leq \eta_1\phi(v,v)$. Since the maximum of $\phi(v,v)$ is η_1, the largest the minimum $\eta_3\phi(v,v)$ can be is $\eta_3\eta_1 = 1$. Since the minimum of $\phi(v,v)$ if η_3, the smallest the maximum $\eta_1\phi(v,v)$ can be is $\eta_1\eta_3 = 1$. So, $\omega_1 \leq \phi(v,v)\phi(w,w) \leq \omega_2$, where $\omega_1 \leq 1$ and $\omega_2 \geq 1$. Since $\phi(v,v)\phi(w,w)$ is a real continuous function, it attains every value between and including its global

minimum and global maximum. Since 1 is between the global minimum of $\phi(v,v)\phi(w,w)$ and the global maximum of $\phi(v,v)\phi(w,w)$, by the Intermediate Value Theorem, there exists w such that $\phi(v,v)\phi(w,w) = 1$. Therefore, since v was chosen arbitrarily, for all v in V, $v \neq 0$, there exists w in V such that $\kappa(span\{v,w\}) = 1$. □

Conjecture 1. *For 9, for arbitrary v in V, $v \neq 0$, $w = ae_1 + be_2 + ce_3$, where*

$$g = \frac{\sqrt{((2bcf(1 - \frac{\eta_2}{\eta_1}))^2 - 4(b^2(\frac{\eta_2}{\eta_1} - 1))(a^2f^2(\eta_1\eta_2 - 1) + 2abdf(1 - \eta_1\eta_2) + b^2d^2(\eta_1\eta_2 - 1) + c^2f^2(\frac{\eta_2}{\eta_1} - 1)) + 2 \cdots}}{2b^2(\frac{\eta_2}{\eta_1} - 1)}$$

Proof. Note: This proof is incomplete.

$$\begin{aligned}
\kappa(span\{v,w\}) &= \frac{R(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\
&= \frac{R_\phi(v,w,w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\
&= \frac{\phi(v,v)\phi(w,w) - \phi(v,w)\phi(w,v)}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle\langle w,v\rangle} \\
&= \frac{\phi(ae_1+be_2+ce_3,ae_1+be_2+ce_3)\phi(de_1+fe_2+ge_3,de_1+fe_2+ge_3) - \phi(ae_1+be_2+ce_3,de_1+fe_2+ge_3)\phi(de_1+fe_2+ge \cdots}{\langle ae_1+be_2+ce_3,ae_1+be_2+ce_3\rangle\langle de_1+fe_2+ge_3,de_1+fe_2+ge_3\rangle - \langle ae_1+be_2+ce_3,de_1+fe_2+ge_3\rangle\langle de_1+fe_2+ge_3,a \cdots} \\
&= \frac{(a^2\phi(e_1,e_1)+b^2\phi(e_2,e_2)+c^2\phi(e_3,e_3))(d^2\phi(e_1,e_1)+f^2\phi(e_2,e_2)+g^2\phi(e_3,e_3)) - (ad\phi(e_1,e_3)+bf\phi(e_2,e_2)+cg\phi(e_3,e \cdots}{(a^2\langle e_1,e_1\rangle+b^2\langle e_2,e_2\rangle+c^2\langle e_3,e_3\rangle)(d^2\langle e_1,e_1\rangle+f^2\langle e_2,e_2\rangle+g^2\langle e_3,e_3\rangle) - (ad\langle e_1,e_1\rangle+bf\langle e_2,e_2\rangle+cg\langle e_3,e_3\rangle)(\cdots} \\
&= \frac{(a^2\eta_1+b^2\eta_2+c^2\eta_3)(d^2\eta_1+f^2\eta_2+g^2\eta_3) - (ad\eta_1+bf\eta_2+cg\eta_3)^2}{(a^2+b^2+c^2)(d^2+f^2+g^2) - (ad+bf+cg)^2} \\
&= \frac{(a^2\eta_1+b^2\eta_2+\frac{c^2}{\eta_1})(d^2\eta_1+f^2\eta_2+\frac{g^2}{\eta_1}) - (ad\eta_1+bf\eta_2+\frac{cg}{\eta_1})^2}{(a^2+b^2+c^2)(d^2+f^2+g^2) - (ad+bf+cg)^2}.
\end{aligned}$$

If $b = 0$, then v is in $span\{e_1, e_3\}$). Since $R(e_1, e_3, e_3, e_1) = 1$, by 5, $\kappa(span\{e_1, e_3\}) = 1$, which by 4 means there exists w in $span\{e_1, e_3\}$ such that $\kappa(span\{v, w\}) = 1$. So, suppose $b \neq 0$. Since $\eta_1\eta_3 \neq \eta_2\eta_3$, $\eta_2 \neq \eta_1$, so $\frac{\eta_2}{\eta_2} \neq 1$, so $\frac{\eta_2}{\eta_2} - 1 \neq 0$. Let

$$g = \frac{\sqrt{((2bcf(1 - \frac{\eta_2}{\eta_1}))^2 - 4(b^2(\frac{\eta_2}{\eta_1} - 1))(a^2f^2(\eta_1\eta_2 - 1) + 2abdf(1 - \eta_1\eta_2) + b^2d^2(\eta_1\eta_2 - 1) + c^2f^2(\frac{\eta_2}{\eta_1} - 1)) + 2 \cdots}}{2b^2(\frac{\eta_2}{\eta_1} - 1)}$$

Then,

$$\begin{aligned}
\frac{(a^2\eta_1+b^2\eta_2+\frac{c^2}{\eta_1})(d^2\eta_1+f^2\eta_2+\frac{g^2}{\eta_1}) - (ad\eta_1+bf\eta_2+\frac{cg}{\eta_1})^2}{(a^2+b^2+c^2)(d^2+f^2+g^2) - (ad+bf+cg)^2} &= \frac{(a^2\eta_1+b^2\eta_2+\frac{c^2}{\eta_1})(d^2\eta_1+f^2\eta_2+\frac{((2bcf(1-\frac{\eta_2}{\eta_1}))^2-4(b^2(\frac{\eta_2}{\eta_1}-1))(a^2f^2(\eta_1\eta_2-1 \cdots}{\cdots}}{(a^2+b^2+c^2)(d^2+f^2+\frac{((2bcf(1-\frac{\eta_2}{\eta_1}))^2-4(b^2(\frac{\eta_2}{\eta_1}-1))(a^2f^2(\eta_1\eta_2-1)+ \cdots}{\cdots}} \\
&=
\end{aligned}$$

The proof remains incomplete.

□

5 Conclusion

The study of constant vector curvature is a new field of inquiry in the realm of curvature conditions, with many intriguing questions to be asked and answered.

Our results use the Ricci tensor and the algebraic curvature tensor to classify when a model space in dimension three with positive definite inner product has constant vector curvature. A unique characteristic of this project is that it involves an algebraic study of a geometric object. Other cases for investigation by this method include model spaces in dimension four or higher, model spaces with nondegenerate inner product, and model spaces in dimension three with positive definite inner product which have a noncanonical algebraic curvature tensor.

6 Open Questions

1. Is constant vector curvature well-defined when $dim(V) \geq 4$? When $\langle \cdot, \cdot \rangle$ is nondegenerate?

2. When $dim(V) = 3$, the Ricci tensor completely characterizes the algebraic curvature tensor of a model space, and so its eigenvalues enable a classification of constant vector curvature. To what degree do the eigenvalues of the Ricci tensor enable a classification of constant vector curvature when $dim(V) \geq 4$? When $\langle \cdot, \cdot \rangle$ is nondegenerate?

3. Assume $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R_\phi \pm R_\psi)$, where $dim(V) = 3$ and $\langle \cdot, \cdot \rangle$ is positive definite. When does $\mathcal{M}$ have $cvc(\epsilon)$?

4. Assume $\mathcal{M} = (V, \langle \cdot, \cdot \rangle, R)$, where $dim(V) = 3$ and $\langle \cdot, \cdot \rangle$ is positive definite. If $\mathcal{M}$ has $cvc(0)$, then does $ker(R) \neq \{0\}$?

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