The Cyclic Cutwidth of Q_n

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Abstract

In this article the cyclic cutwidth of the n-dimensional cube is explored. It has been conjectured by Dr. Chavez and Dr. Trapp that the cyclic cutwidth of Q_n is minimized with the Greycode numbering. Several results have been found toward the proof of this conjecture.

1 Introduction

Let $G = (V, E, \partial)$ represent a graph with a set, V, of vertices, a set, E, of edges, and a function $\partial : E \to \binom{V}{2}$ which identifies the two distinct vertices incident to each edge. G has often been analogised to an electric circuit in the literature.
A numbering of the vertices, η, is a function that assigns a distinct number from 1 to m to each of the vertices in G, where $m = |V|$. A numbering can most naturally be thought of as an embedding of G into a linear chassis, though other host graphs may be considered. The main emphasis of this paper will be with a circular host graph. To distinguish these two host graphs, the letters 'l' and 'c' will be used as a prefixes for linear and cyclic, respectively.
There are three major properties of an embedding of a graph: *bandwidth* (*bw*), *wirelength* (*wl*), and *cutwidth* (*cw*) [4].

$$lbw(G, \eta) = max\{|\eta(v) - \eta(w)| : \{v, w\} \in E\}.$$

That is, *lbw* is the maximum distance between two vertices connected by an edge. For the graph, $lbw(G)$ is the minimum of these $lbw(G, \eta)$'s over all numberings.

$$lwl(G, \eta) = \sum_{\{v,w\} \in E} |\eta(v) - \eta(w)|.$$

That is, *lwl* is the sum of the lengths of all the edges. $lwl(G)$ is the minimum, again, over all numberings.

$$lcw(G,\eta) = \max_{l} |\{\{v,w\} \in E : \eta(v) \leq l < \eta(w)\}|.$$

That is, *lcw* is the maximum number of edges that pass between two consecutively numbered vertices. $lcw(G)$ is the minimum of these maxima over all numberings. *cbw*, *cwl*, and *ccw* are defined similarly to their linear counterparts, with the appropriate adjustments made, particularly, vertices are numbered congruence classes instead of numbers. In this paper only *wl* and *cw* are of interest. **Note:** For *cbw*, *cwl*, and *ccw* there are two choices for which direction an edge should go. For *cbw* and *cwl* we clearly only want to choose the direction that minimizes the length of the edge. However, there are graphs with numberings that have a smaller *ccw* when an edge goes the long way around.
In finding the values of $cw(G)$, $wl(G)$ and $bw(G)$, a useful function from the area of discrete isoperimetric problems, the theta function, can be used. This function will be limited in its use here as follows:

$$\theta(S) = |\{v,w\} \in E, v \in S, w \notin S|$$

$$\text{and } \theta(l) = \min_{|S|=l} \theta(S).$$

In other words, $\theta(S)$, $S \subseteq V$ is the number of edges that have exactly one vertex in S, and over all sets $S \subseteq V$ of size l, $\theta(l)$ is the least number of edges that have exactly one vertex in the set. The notation $\theta_n(S)$ and $\theta_n(l)$ will be used when G is Q_n. For clarity, lowercase letters will be used to represent numbers and uppercase letters will be reserved for sets.
Finding the value of $cw(G)$ is called the *cutwidth problem.* The cutwidth problem is NP-complete for graphs in general [4]. However, the solution to the cutwidth problem is know for special cases like an n-dimensional cube (Q_n) embedded on linear and grid host graphs. (When the host graph is a grid, the term *congestion* is used instead of cutwidth.) A conjecture has been made for $cw(Q_n)$ when the host graph is a circle, called the CT conjecture.
The CT conjecture (named after Chavez and Trapp) states that the Greycode numbering gives $ccw(Q_n)$. Or, as a formula, the CT conjecture asserts $ccw(Q_n) = \lfloor \frac{5 \cdot 2^{n-2}}{3} \rfloor$ when $n \geq 2$. The Greycode numbering is recursively defined with a base case in Q_2. Q_n is two copies of Q_{n-1}, so if we know the numbering for Q_{n-1}, that numbering is copied in reverse on the second Q_{n-1} to give the numbering for Q_n. One feature of the Greycode numbering is that consecutively numbered vertices are adjacent to each other.

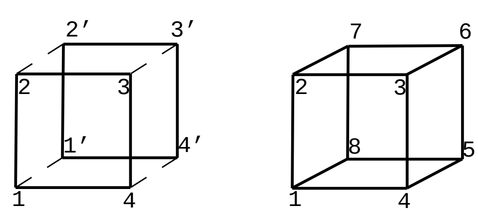

Diagram 1: Generating the Greycode numbering for Q_3 from Q_2.

After some historical background, there is a section summing up many of the important results, several of which are unpublished, pertaining to $ccw(Q_n)$. The fourth section presents the results from Ching J. Guu's Ph.D. dissertation [5], which are used in the new results of the fifth section.

2 Historical background

Before 1996 $ccw(Q_3)$ was already known from using exhaustive searching by computer. In [7] Beatrice James found an alternate method of determining $ccw(Q_n)$ and applied it to Q_3 and to Q_4. Her method is extendable to higher dimension cubes, however, the number of cases blows up. In 2001 Ryan Aschenbrenner used another method [1] to find $ccw(Q_5)$. His method is also extendable, but has similar problems. Currently, Candi Castillo [3] is using Aschenbrenner's method to prove $ccw(Q_6)$. So far as has been tested, the CT conjecture has held.

During this 1996-present period other advances have been made with variations on the $ccw(Q_n)$ problem. In 1997 Ching J. Guu claimed in [5] that $cwl(Q_n)$ is minimized with the Greycode numbering. In 2000 Bezrukov et al. [2] published their proof that $lcw(Q_n)$ is minimized with a lexicographic numbering. In the proof they used an equivalent discrete isoperimetric problem. Also proved in that paper is the congestion of Q_n, which is closely connected to $lcw(Q_n)$. In [6] the Hale's numbering is shown to minimize $lwl(Q_n)$.

3 Important results

3.1 $ccw(Q_n)$–Methods

James's method [7] is based on the fact that as far as cuts are concerned, there are only two ways of looking at each disjoint Q_2. A Q_2 can contribute one cut all the way around the cycle or two cuts in a local region. This method can be extended in two ways–simply increase the cases as Q_n goes up in dimension or increase the size of the subcubes. The second way increases the number of cases because the number of ways of representing larger subcubes increases. It appears that neither extension is suitable for proving $ccw(Q_n)$.

Aschenbrenner's method [1] involves a diameter that cuts the cycle in two pieces and a pair of disjoint Q_{n-1}'s that are split by the diameter. One can look at the size of the split given a particular pair of Q_{n-1}'s with respect to a particular diameter. The split size is the number of vertices from each Q_{n-1} that are on one side of the diameter. However, more than one diameter may be considered. In fact, the diameter is free to move around the cycle, and the choice of Q_{n-1}'s is free as well, so long as they are complementary Q_{n-1}'s.

In looking at a single diameter, Aschenbrenner developed a useful notation to help find cutwidth. His use of the notation is for Q_{n-1}'s, though it can be extended to consider any pair of complementary subgraphs:

$$\left[\begin{array}{c|c|c} |A| & |e_{AB}| & |B| \\ \hline |e_{AC}| & |e_{AD}|+|e_{BC}| & |e_{BD}| \\ \hline |C| & |e_{CD}| & |D| \end{array}\right]$$

In the most general sense of the notation, vertex set $A \cup C$ is that of one subgraph, and vertex set $B \cup D$ is the set for the other subgraph. The first row indicates what is on one side of the diameter. The other side of the diameter is indicated by the third row. The middle row counts the number of edges that cross the diameter. The number of edges that have one vertex in X and the other vertex in Y is $|e_{XY}|$.
In Aschenbrenner's paper it is mentioned that when there is a 5/11 split (or 11/5 split, depending on which Q_4 you count first and on which side of the cycle you count) the problem was easy for Q_5. Generally, if there is a $\frac{2}{3}/\frac{1}{3}$ split the problem is solved. More precisely, a $\frac{2^n+(-1)^{n+1}}{3}/\frac{2^{n-1}+(-1)^n}{3}$ split is an easy split for Q_n.

3.2 Proof for a $\frac{2}{3}/\frac{1}{3}$ split

Theorem 1: When there is at least a $\frac{2^n+(-1)^{n-1}}{3}/\frac{2^{n-1}+(-1)^n}{3}$ split, the largest cut is at least $\frac{5\cdot 2^{n-2}-1}{3}$ when n is odd or $\frac{5\cdot 2^{n-2}-2}{3}$ when n is even.
Lemma: When there is a split greater than x/y, an x/y split also occurs.
Proof of Lemma: With no loss in generality we can assume $x \geq y$. Let $k \geq 0$. If an $x+k/y-k$ split exists, the diameter can be rotated one vertex pair at a time, 2^{n-1} times. At that point, the diameter will be in the same position but oppositely oriented to its original position, giving a $y-k/x+k$ split. With each rotation the left side of the split can increase by 1, decrease by 1, or stay the same. Since $y-k \leq x \leq x+k$, the left side of the split must have been x at some point. Thus, an x/y split exists.
Proof of Theorem 1: From the lemma, only a $\frac{2^n+(-1)^{n-1}}{3}/\frac{2^{n-1}+(-1)^n}{3}$ split has to be proven. Using Aschenbrenner's notation, this split is written:

$$\left[\begin{array}{c|c|c} \frac{2^n+(-1)^{n+1}}{3} & \frac{2^{n-1}+(-1)^n}{3} & \frac{2^{n-1}+(-1)^n}{3} \\ \hline \theta_{n-1}(\frac{2^{n-1}+(-1)^n}{3}) & \frac{2^{n-1}+2(-1)^{n+1}}{3} & \theta_{n-1}(\frac{2^{n-1}+(-1)^n}{3}) \\ \hline \frac{2^{n-1}+(-1)^n}{3} & \frac{2^{n-1}+(-1)^n}{3} & \frac{2^n+(-1)^{n+1}}{3} \end{array}\right].$$

Each vertex in each Q_{n-1} is connected by an edge to one vertex in the other Q_{n-1}. So $\frac{2^{n-1}+(-1)^n}{3}$ edges connecting Q_{n-1}'s is the maximum that can stay on each side of the diameter. This minimizes the number of edges between Q_{n-1}'s that go through the diameter (See Diagram 2 and accompanying example).

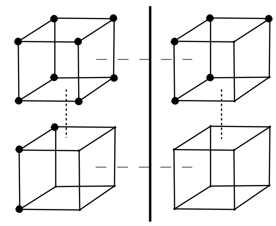

Diagram 2: Q_5 as $Q_3 \times Q_2$. Solid vertices represent vertices on one side of the diameter. Q_4's are separated by the solid line.

In Aschenbrenner's notation Diagram 2 would be represented:

$$\left[\begin{array}{ccc} 11 & 5 & 5 \\ \hline 10 & 6 & 10 \\ \hline 5 & 5 & 11 \end{array}\right].$$

Hence, the minimum number of edges that cross the diameter is $2 \cdot \theta_{n-1}\left(\frac{2^{n-1}+(-1)^n}{3}\right) + \frac{2^{n-1}+2(-1)^{n+1}}{3}$, and $ccw(Q_n)$ is at least half of this. In [2] a recursion is given for finding $\theta_n(l)$:

$$\theta_n(l) = \begin{cases} 2l + \theta_{n-2}(l) & \text{if } 0 \le l \le 2^{n-2} \\ 2^{n-1} + \theta_{n-2}(l - 2^{n-2}) & \text{if } 2^{n-2} \le l \le 2^{n-1} \end{cases}$$

Two cases arise–one when n is odd, and one when n is even.

Case I. When n is odd

$$\begin{aligned} ccw(Q_n) &\ge \theta_{n-1}\left(\frac{2^{n-1}-1}{3}\right) + \frac{2^{n-2}+1}{3} \\ &= \theta_{n-1}(1 + 2^2 + 2^4 + \cdots + 2^{n-3}) + \frac{2^{n-2}+1}{3} \\ &= 2\left(\frac{2^{n-1}-1}{3}\right) + \frac{2^{n-2}+1}{3} \\ &= \frac{5 \cdot 2^{n-2} - 1}{3}. \end{aligned}$$

This is the same as $ccw(Q_n$, Greycode) for odd n.

Case II. When n is even

$$\begin{aligned} ccw(Q_n) &\ge \theta_{n-1}\left(\frac{2^{n-1}-2}{3} + 1\right) + \frac{2^{n-2}-1}{3} \\ &= \theta_{n-1}(1 + 2^1 + 2^3 + \cdots + 2^{n-3}) + \frac{2^{n-2}-1}{3} \\ &= \frac{2^n - 1}{3} + \frac{2^{n-2}-1}{3} \\ &= \frac{5 \cdot 2^{n-2} - 2}{3}. \end{aligned}$$

This is the same as $ccw(Q_n$, Greycode) for even n. $\square$

4 Cyclic wirelength of Q_n

In Ching J. Guu's Ph.D. dissertation [5], a proof for $cwl(Q_n)$ is claimed. Her claim is that the Greycode numbering minimizes cyclic wirelength. En route, she creates a derived network to convert the problem into a discrete isoperimetric problem. Then she defines $Type(S)$, $S \subseteq V_n$. There are $2n$ Q_{n-1}'s in Q_n, which are denoted H_i.

$$Type(S) = \min_{1 \leq i \leq 2n} |S \cap H_i|.$$

That $Type(S)$ and the size of a split are related is not immediately obvious, so a more formal (and slightly restrictive) definition for split size will be used in this section. For $S \subseteq V_n$, the size of the split of S is,

$$Split(S) = \max_{1 \leq i \leq 2n} |S \cap H_i|.$$

This definition gives only the bigger side of the split, but allows the spliting line to be a nondiameter. Now it should be fairly evident that $Type(S) + Split(S) = |S|$, and that consequently, the splits used earlier are $Type(S)/Split(S)$ or $Split(S)/Type(S)$, depending on the orientation of the splitting line.

4.1 *big* and *small*

When $|S| = 2^{n-1}$, $0 \leq Type(S) \leq 2^{n-2}$. Guu abbreviates $Type(S)$ with t, and calls a set *big* if $t \geq 2^{n-3}$ and *small* if $t \leq 2^{n-3}$. When a path in the derived network goes from a set to its complement, if all the sets in the path are *small*, the Greycode numbering is shown to minimize $cwl(Q_n)$. When a set, S', in the path is of *big* type, Guu claims that $\theta_n(S') \geq \frac{3}{4} \cdot 2^n$, which is large enough to not need any further consideration. Her proof of this inequalty, however, contains at least one error. It has not been determined how grave the error is. Since the approach may have some utility, the outline is included here.
First, $f(x) = \frac{3}{4} - \frac{64}{7}(x - \frac{1}{2})^2$ is introduced, which has the property that $f(x-t) + f(x+t) + 2t \geq 2f(x)$ when $0 \leq t \leq \frac{7}{64}$. Next, it is demonstrated that when $\frac{2^n}{24} \leq Type(S) \leq (\frac{1}{24} + \frac{7}{64})2^n$ and $|S| \leq 2^{n-1}$, $\theta_n(S) \geq f(x) \cdot 2^n$. The third and final step is that when $Type(S) \geq (\frac{1}{24} + \frac{7}{64})2^n$, $\theta_n(S) \geq f(x) \cdot 2^n$.
An error occurs in the proof of the final step. She takes $S_1 \cup S_2 = S$, with $S_1 \cap S_2 \neq \{\}$. Then she continues $|S_1| + |S_2| = |S|$.

5 New results

5.1 A new lower bound

Up to this point, the best lower bound known for $ccw(Q_n)$ was $ccw(Q_n) \geq \frac{1}{2} lcw(Q_n)$. We also know an upper bound for $ccw(Q_n)$ (which is sharp at least up to $n = 6$) is $ccw(Q_n) \leq \lfloor \frac{5}{8} lcw(Q_n) \rfloor$. (This formula is incorrect for the trivial cases of $n \leq 1$, when no cycles exist.) For this section it

will be assumed that Guu's results are correct, and that the Greycode numbering minimizes the cyclic wirelength of Q_n. With this assumption, a larger lower bound can be calculated. Before calculating this new lower bound we first must know what the value of $cwl(Q_n)$ is.
Claim: $cwl(Q_n) = 2^{2n-2} + 2^{2n-3} - 2^{n-1}$
Proof of Claim: This claim will be proven by summing the lengths of each wire in the Greycode numbering. Starting with the horizontal and vertical wires, there are 4 each of wires with length $1, \ldots, 2^{n-3}, 2^{n-1}$. There are 2^2 groups of edges along primary diagonals whose wires have lengths $1, \ldots, 2^{n-2}-3, 2^{n-2}-1$. There are 2^3 groups of edges along secondary diagonals whose wires have lengths $1, \ldots, 2^{n-3}-3, 2^{n-3}-1$, etc.

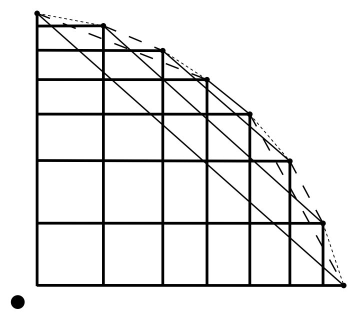

Diagram 3: One quarter of a Q_5 Greycode.

For example, Diagram 3 depicts a quarter of a Q_5 Greycode. The wires along the primary diagonal are solid, the wires along the secondary diagonals are dashed, and the wires along the tertiary diagonals are dotted. The dot in the lower left corner is the location of the center of the cycle. Keep in mind, the horizontal and vertical wires have been truncated in the diagram, and that there are four times as many copies of each of the diagonal wires.
The grand sum of lengths of wires for Q_n is thus:

$$\begin{aligned} cwl(Q_n) &= 4[(2^{n-1}-1) + (2^{n-1}-3) + \cdots + 1] + \\ &\quad 2^2[(2^{n-2}-1) + (2^{n-2}-3) + \cdots + 1] + \\ &\quad 2^3[(2^{n-3}-1) + (2^{n-3}-3) + \cdots + 1] + \cdots + 2^{n-1} \\ &= 4(2^{2n-4}) + 2^2(2^{2n-6}) + 2^3(2^{2n-8}) + \cdots + 2^{n-1} \\ &= 2^{2n-2} + 2^{2n-4} + 2^{2n-5} + \cdots + 2^{n-1} \\ &= 2^{2n-2} + 2^{2n-3} - 2^{n-1} \end{aligned}$$

Therefore, $ccw(Q_n) \geq \frac{cwl(Q_n)}{2^n} = 2^{n-2} + 2^{n-3} - \frac{1}{2}$. Or, in terms of $lcw(Q_n)$: $ccw(Q_n) \geq \lfloor \frac{9}{16} lcw(Q_n) \rfloor$.

5.2 Open conjectures

Reverse engineering this process starting with a desired lower bound gives the following:

$$cwl(Q_n) > \begin{cases} 2^n\left(\frac{5\cdot 2^{n-2}-1}{3}-1\right) & \text{if } n \text{ is odd} \\ 2^n\left(\frac{5\cdot 2^{n-2}-2}{3}-1\right) & \text{if } n \text{ is even.} \end{cases}$$

It would be nice to find this as the lower bound to $cwl(Q_n)$ for when a $\frac{2}{3}/\frac{1}{3}$ split does not exist. If it is a lower bound, the CT conjecture would be verified. This can be achieved if Guu's method can be applied with $f(x) = \frac{5}{6} - k(x-\frac{1}{2})^2$ and $Type(S) \geq \frac{1}{3}2^{n-1}$ (as opposed to her $f(x) = \frac{3}{4} - \frac{64}{7}(x-\frac{1}{2})^2$ and $Type(S) \geq 2^{n-3}$).

6 Conclusion

When a $\frac{2}{3}/\frac{1}{3}$ split exists, the Greycode numbering optimizes $ccw(Q_n)$. The Greycode also is likely the optimal numbering for $cwl(Q_n)$, though there remain holes in the proof. If the Greycode does optimize $cwl(Q_n)$, the cyclic cutwidth problem is "cut in half." A stronger version of the cyclic wirelength problem could solve the cyclic cutwidth problem.

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