Volume of Hyperbolic Closed 3-Braids

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Abstract

The goal of this paper is to improve on Marc Lackenby's upper bound on the volume of hyperbolic, closed 3-braids.

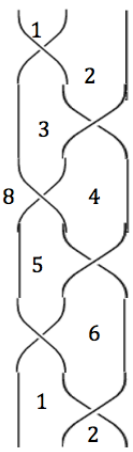
Introduction

The volume of a hyperbolic, closed 3-braid is defined to be the volume of the complement of the knot. The complement of a knot is often thought of as a knot living in a solid ball, with a hollow torus following the knot. The volume of a knot is a knot invariant, although mutants of knots all have the same volume, therefore the volume canknot always differentiate between distinct knots. The hyperbolic knot known to have the smallest volume is the figure 8 knot with volume 2.02988..., made up of 2 ideal tetrahedra.

Marc Lackenby stated the hyperbolic volume of any closed 3-braid is less than or equal to $v_3(16t(D) - 16)$ where v_3 is the maximum volume of an ideal tetrahedra, estimated to be 1.01494 and $t(D)$ is the twist number of the braid. In the appendix of Lackenby's *The Volume of Hyperbolic Alternating Link Complements*, Ian Agol and Dylan Thurston modify Lackenby's result, stating the upper bound for the volume is $10v_3(t(D) - 1)$.

Upper Bounds on Volume

Figure 1:



Lemma: *The number of triangles that make up the component of a $(\sigma_1\sigma_2^{-1})^k$ knot with $k \geq 3$ is less than or equal to $4k-4$.*

Proof. (by induction)

<u>Base Case:</u> $k = 3$

By examining the braid diagram of $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$, it is evident that there are 8 triangles.

Note that the braid is closed, so the top two regions connect with the bottom two, making 2 triangles.

Since $4k = 4 \cdot 3 = 12$ and $8 \leq 12 - 4 = 8$, we have finished the base case. □

<u>Successor Case:</u> $k = n + 1$

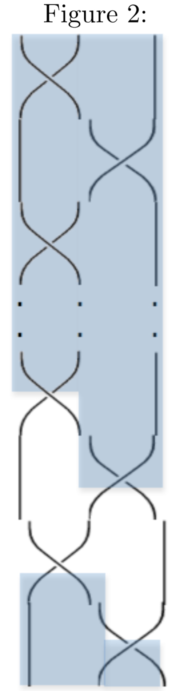
Assume $(\sigma_1\sigma_2^{-1})^n \leq 4n - 4$.

We want to show $(\sigma_1\sigma_2^{-1})^{n+1} \leq 4(n+1) - 4 = 4n$.

The diagram to the right is of $(\sigma_1\sigma_2^{-1})^{n+1}$ where the blue rectangle is showing what is contained in $(\sigma_1\sigma_2^{-1})^n$.

Counting the interior triangles of the braid is straightforward, with 2 more interior triangles in $(\sigma_1\sigma_2^{-1})^{n+1}$ than $(\sigma_1\sigma_2^{-1})^n$.

Figure 2:



Note again that the braid is closed, therefore the bottom two regions connect to the top two, making triangles that are already accounted for in $(\sigma_1\sigma_2^{-1})^n$.

The $(\sigma_1\sigma_2^{-1})^n$ braid has n vertices on each side, which can be divided into $n-2$ triangles on each side.

The $(\sigma_1\sigma_2^{-1})^{n+1}$ braid has $n+1$ vertices on each side, which can be divided into $n-1$ triangles on each side.

So, there are two more exterior triangles in $(\sigma_1\sigma_2^{-1})^{n+1}$ than $(\sigma_1\sigma_2^{-1})^n$.

Since there are 2 more exterior and 2 more interior triangles, altogether there are 4 more triangles in $(\sigma_1\sigma_2^{-1})^{n+1}$ than $(\sigma_1\sigma_2^{-1})^n$.

Therefore, $(\sigma_1\sigma_2^{-1})^{n+1} \leq 4n - 4 + 4 = 4n$, and the successor case holds. □ ■

Lemma: *The number of tetrahedra in the upper half of the complement of $(\sigma_1\sigma_2^{-1})^k$ is less than or equal to $3k-5$.*

Proof.

Consider $(\sigma_1\sigma_2^{-1})^k$.

Since there are k crossings on either side, the exterior regions of the braid are k-gons, which can be split into $k-2$ triangles.

We will triangulate the k-gon by connecting the first vertex to the 3rd, 4th, $\ldots$, $(k-1)$th vertex.

Before triangulation, the k-gon, the first vertex has valence 4. After triangulation, the first vertex has valence $4 + [(k-1) - 2] = k + 1$.

Now, we are going to put the first vertex at infinity, and cone from it.

Since it has valence $k+1$, that implies that there are $k+1$ triangles connecting the sides of the base to the vertex at infinity.

By the previous lemma, we know there are $4k - 4$ triangles total.

Since there are $k + 1$ on the sides, that leaves $4k - 4 - (k + 1) = 3k - 5$ triangles in the base.

The triangles in the base all connect to the point at infinity, forming tetrahedra, and hence there are $3k - 5$ tetrahedra. ■

References

1. M. Lackenby, *The Volume of Hyperbolic Alternating Link Complements*