Linking Number of a Linear Embedding with Two Components

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Abstract

The linking number, the crossing number, and stick number of a linear embedding with 2V-vertices is defined. The linking number of any graph with two components on 2V-vertices is shown to have a maximum related to the number of vertices on the graph. This also allows the crossing number to have a minimum.

1 Introduction

1.1 Linear Embeddings

A linear embeddings of K_{2V} in $\mathbb{R}^3$ means that each line segment is a linear straight line, and that each line segment connects two vertices. Since we are working with knots, the linear embedding must be closed, which means that if you start at any one point on the graph and follow the line segments in one continuous direction, you will end at the point at which you started. It is important to remember that the intersection of two line segments on a graph does not create a new vertex, rather, it is considered to be a crossing, which is defined in the next section.

1.2 Linking Number and Crossing Number

The crossing number of a knot cr(K_n), where n is the number of vertices, in $\mathbb{R}^3$ is defined to be the minimum number of crossings in any diagram.

Two components of a knot are said to cross if the edge of one knot passes over the other like so:
(insert classy picture of edges crossing here)
The crossing number of the unknot is defined to be:

$$\text{cr(unknot)}= 0$$

The linking number $\sigma(cr)$ of a two component link on a linear embedding $K_n$in $\mathbb{R}^3$, where n is the number of vertices, is defined to be:

$$\tfrac{1}{2}\sum\sigma(cr)$$

where c is a crossing between two components.
Two components are said to be linked if it is not possible to separate the two components without deleting or breaking one of the edges of one component.

The linking number of the unknot is defined to be:

$$\sigma(\text{unknot}) = 0$$

1.3 Stick Number

The stick number of a knot S(K_n)in $\mathbb{R}^3$, where n is the number of vertices, is the fewest number of sticks needed to create it.

For instance, the stick number of the unknot is defined to be:

$$\text{S(unknot)} = 3$$

and the stick number of the trefoil is defined to be:

$$\text{S(trefoil)} = 6$$

2 Maximizing Linking Number between a Triangle and an Additional Component

Lemma: Let K_{2V} in $\mathbb{R}^3$ be our favorite linear embedding and (x, y, z) be any triangle on K_{2V}, with C being the other component using the remaining vertices.

$$\text{Prove: } \sigma(C, (x, y, z)) \leq \frac{2V}{3} - 1$$

Proof: Each edge has two vertices and two edges are requred in a crossing.

Let a and be be any two vertices on our embedding K_{2V}.

Let d and c (Without loss of generality) represent any of the two vertices in triangle (x, y, z).

Then we have two edges whose linking number can be determined by the ordering of the vertices a, b, c and d since there are only 4! ways to arrange them.

Case 1: Let the ordering of our vertices be (a,b,c,d) where $a < b < c < d$. In other words, the max of edge ab, is less than the min of edge cd, and the max of edge cd, is greater than the max of ab. Then the edges clearly do not cross, so therefore do not contribute to linking number when ordered this way. Similarly is the case for $b < c < d < a$, $c < d < a < b$ and $d < a < b < c$.

Case 2: Let the ordering of our vertices be (a,b,d,c) where $a < b < d < c$. In other words, the max of ab, is less than the min of edge dc, and the max of dc, is greater than the max of ab. Then the edges clearly do not cross, so therefore do not contribute to linking number when ordered this way. Similarly is the case for $b < d < c < a$, $c < a < b < d$ and $d < c < a < b$.

Case 3: Let the ordering of our vertices be (a,c,d,b) where $a < c < d < b$. In other words, the min of cd is less than the min of ab and also less than the max of ab. Then the edges clearly do not cross, so therefore do not contribute to linking number when ordered this way. Similarly is the case for $b < a < c < d$, $c < d < b < a$ and $d < b < a < c$, which are all cycles of the original cycle (a,c,d,b).

Case 4: Let the ordering of our vertices be (a,d,c,b) where $a < d < c < b$. In other words, the min of dc is less than the min of ab and also less than the max of ab. Then the edges clearly do not cross, so therefore do not contribute to linking number when ordered this way. Similarly is the case for $b < a < d < c$, $c < b < a < d$, and $d < c < b < a$, which are all cycles of the original cycle (a,c,d,b).

It is clear that these first four cases do not contribute to the linking number since none of them cross. However, there remain two more cases which result in crossings.

Case 5: Let the ordering of our vertices be (a,c,b,d) where $a < c < b < d$. In other words, the min of cd is greater than the min of ab, and the max of ab is less than the max of cd. This creates a type of betweenness, causing the edges to cross. This is similarly the case for $b < d < a < c$, $c < b < d < a$, and $d < a < c < b$, all of which are cycles of our initial form. It should be noted, however, that $a < c < b < d$ and $b < d < a < c$ crossings result in $\sigma(cr) = -1$ and that $c < b < d < a$ and $d < a < c < b$ produce $\sigma(cr) = +1$.

Case 6: Let the ordering of our vertices be (a,d,b,c) where $a < d < b < c$. In other words, the min of dc is greater than the min of ab, but the max of ab is less than the max of dc. This creates a type of betweenness, causing the edges to cross. This is similary the case for $b < c < a < d$, $c < a < d < b$, and $d < b < c < a$, all of which are cycles of our initial form. It should be noted, however, that $a < d < b < c$ and $b < c < a < d$ produce $\sigma(cr) = +1$ and that $c < a < d < b$ and $d < b < c < a$ result in $\sigma(cr) = -1$.

Now if we go back to our embedding of triangle (x,y,z) and our other component C, we know the only types of orderings we have to consider are found

in Case 5 and Case 6. suppose that we have $2V$ vertices total and that triangle (x,y,z) takes any three vertices on our embedding. This leaves us with a total of $2V-3$ vertices for component C. Suppose that our triangle is set up like this: center(insert snazzy picture of triangle with the groups of vertices)
Where J, M, N and P are groups of remaining vertices and may or may not be equal to one another such that $J+M+N+P+3=2V$ and where J represents all vertices less than x, M represents all vertices less than y but greater than x, N represents all vertices greater than y but less than z, and P represents all vertices greater than z.

By the previous definition of linking number and the first four cases, we know if an edge is to cross triangle (x,y,z) it must go from one group of vertices to another (i.e. If a vertex j_1 in J, is to have an edge which crosses the triangle (x,y,z) and create a potential for linking, it must connect to a vertex in M, N or P.

idea leaves us with another three cases to consider.

Case 1: Say we are considering two edges, one with vertices a and b and the other (without loss of generality) with vertices x and y.

Suppose that vertex a is in J and vertex b is in M. **(insert picture!!!)**

It is clear that ab crosses xy, however if you also consider the edge xz, you would notice that:

$$\sigma((a,b),(x,y))+\sigma((a,b),(x,y))=0$$

since both xz and xy are underneath ab, there is no actual linking occuring between these edges. This is also the case if vertex a is in J and vertex b is in N. We can ignore the case of vertex a in J and vertex b is in P, because they would not cross the triangle (x,y,z).

Case 2: Suppose that vertex a is in M and vertex b is in N (case 2i) or P (case 2ii).

2i: b is in N - If you look at the linking number between ab and xy, and ab and zy:

$$\sigma((a,b),(x,y))=-1 \text{ and } \sigma((a,b),(z,y))=-1$$

It is clear that there is a linkage created. We will look further at this case later.

2ii: b is in P - If you look at the linking number between ab and xy, and ab and zy:

$$\sigma((a,b),(x,y))=-1 \text{ and } \sigma((a,b),(z,y))=+1$$

it is clear that the summation of these two numbers is 0, and that since ab is under the triangle, no link is actually created.

Therefore, if the edge ab is to be linked with the triangle (x,y,z) then $x < a < y < b < z$.

Now we must consider the next edge on the component C containing ab. Let us call this next edge bc. There are three possibilities for this edge.

Case 1: If we look back to Case 2i from the previous section, we see that vertex a is in M and vertex b is in N. Suppose that the new vertex c was also in M. Then bc "undoes" the link produced by ab. Causing the components to be no longer linked. Therefore, this case can be disregarded and if the components are to be linked, vertex c is in either J or P.

Case 2: If we look back to Case 2i from the previous section, we see that vertex a is in M and vertex b is in N. Suppose that the new vertex c is in J. Then, as you can see by the picture below, the two components are still linked, because the new edge bc does not undo the linking done by ab.

Case 3: If we look back to Case 2i from the previous section, we see that vertex a is in M and vertex b is in N. Suppose that the new vertex c is in P. Then, as you can see by the picture below, the two components are still linked, because the new edge bc does not undo the linking done by ab.

Now it is clear from the previous cases that if there is to be a link created between two components, they must exhibit the similar characteristics as exhibited in Case 2 and Case 3 from the previous section. From these two cases, we can draw the conclusion that the linking number is maximized if $2V$, our total number of vertices, is divided up evenly among three groups of vertices, as seen and as is well known, that the minimum number required to make a link is three edges in each component. So from this we can draw that if we want to maximize the potential linking number on the remaining vertices $2V - 3$, we should divide by 3, the number of groups of vertices utilized in making a link. Now what we have is three regions, each with an equal amount of vertices: **(INSERT PICTURE HERE!!!)**
and the number of vertices in each region, n, can be defined as:

$$n \leq \frac{2V}{3} - 1$$

and as can be seen from this equation, n is maximized when $2V$ is divisible by 3, which also results in the linking number being maximized and: $\sigma(cr) \leq \frac{2V}{3} - 1$